KSÜ Mühendislik Bilimleri Dergisi, 26(Özel Sayı), 2023 KSU J Eng Sci, 26(Special Issue), 2023
9th Uluslararası IFS Çağdaş Matematik ve Mühendislik Konferansı Özel Sayısı
9th International IFS Contemporary Mathematics and Engineering Conference Special Issue
Araştırma Makalesi Research Article

Kahramanmaras Sutcu Imam University
Journal of Engineering Sciences

Geliş Tarihi : 08.08.2023 Received Date : 08.08.2023
Kabul Tarihi : 29.09.2023 Accepted Date : 29.09.2023

RELATIVE CONTROLLABILITY OF THE φ-CAPUTO FRACTIONAL DELAYED SYSTEM WITH IMPULSES

BAŞKABİR FONKSİYONA BAĞLI CAPUTO KESİRLİ ANİ DEĞİŞİMLİ GECİKMELİ SİSTEMİN GÖRECELİ KONTOL EDİLEBİLİRLİĞİ

Mustafa AYDIN[1] (ORCID: 0000- 0003-0132-9636)

[1] Van Yüzüncü Yıl Üniversitesi, Muradiye Meslek Yüksekokulu, Tıbbi Hizmetler ve Teknikler Bölümü, Van, Türkiye

*Sorumlu Yazar / Corresponding Author: Mustafa AYDIN, m.aydin@yyu.edu.tr

ABSTRACT

The impulsive fractional delayed differential system with the Caputo derivative with respect to another function is considered. An explicit solution to the system in the light of the available studies on this subject is determined and its existence and uniqueness are debated. Lastly, the stability and controllability of the given system are investigated.

Keywords: Existence uniqueness, impulsive fractional delayed system, relative controllability, Ulam-Hyers stability

ÖZET

Herhangi bir fonskiyona göre tanımlanmış Caputo türevli ani değişmeli kesirli gecikmeli bir sistem dikkate alınmaktadır. Bu konuda mevcut çalışmaların ışığında sistemin sarih bir çözümü belirlenmekte ve çözümün varlığı ve tekliği tartışılmaktadır. Son olarak, verilen sistemin kararlılığı ve kontrol edilebilirliği araştırılmaktadır.

Anahtar Kelimeler: Ani değişmeli kesirli gecikmeli sistem, varlık teklik, Ulam-Hyers kararlılığı, göreceli kontrol edilebilirliği

INTRODUCTION

Fractional calculus is regarded as a generalization of integer calculus. Of course, this generalization contributes different positive capabilities which integer calculus does not have to fractional calculus. For example, according to researchers in this field, this enables fractional calculus to model almost all of scientific problem more suitable than integer order, numerical approaches to fractional calculus give better results compared to integer calculus, etc. Fractional calculus begins to be used in many areas such as mathematical physics, biophysics, engineering, signal processing, etc. For more details, all of reference section can be scanned.

A differential equation which consists of the present state and its rate of changes is said to be a delayed differential equation(Aydin et al., 2022; Aydin & Mahmudov, 2022; Mahmudov, 2022; Elshenhab & Wang, 2021b, 2021a; Mahmudov & Aydin, 2021; Liu et al., 2021; Mahmudov, 2019; Mahmudov, 2018; Li & Wang, 2018; Khusainov & Shuklin, 2003) if it also includes the past state. As stated and shown in (Mahmudov, 2019), a solution of a linear system $\rho'(\varsigma) = M\rho(\varsigma), \varsigma \geq 0$ has the form $\rho(\varsigma) = e^{M\varsigma}\rho(0)$, where the exponential matrix is also called fundamental matrix having a simple structure, but, it becomes more complex for seeking a fundamental matrix for a linear delayed system $\rho'(\varsigma) = M\rho(\varsigma) + A\rho(\varsigma - r), \varsigma \geq 0, r > 0$ with an initial condition $\rho(\varsigma) = \vartheta(\varsigma), -r \leq \varsigma \leq 0$, because of its fundamental matrix's complex structure caused by the delay parameter. Its solution, which is obtained by (Khusainov

ToCite: AYDIN, M., (2023). RELATIVE CONTROLLABILITY OF THE φ-CAPUTO FRACTIONAL DELAYED SYSTEM WITH IMPULSES. *Kahramanmaraş Sütçü İmam Üniversitesi Mühendislik Bilimleri Dergisi*, 26(Özel Sayı), 1121-1132.

& Shuklin, 2003) under the assumption of commutativity of the coefficient matrices M and A, naturally has a complex structure. So, it is difficult to work on such an equation according to equivalent studies. When we have look at the literature, these kinds of systems have been investigated in terms of controllability, stability, and existence and uniqueness of solutions (You et al., 2020; Liang et al., 2017; Wang et al., 2017; Khusainov & Shuklin, 2005) of the systems.

Generally, a differential equation is exploited to describe the dynamics of changing processes. The dynamics of many changing phenomena count on abrupt changes such as shocks, natural phenomena which is an observable event that is not non-made. These sorts of processes own short-dated perturbations (deviations) of continuous dynamics. When the duration of the entire advancement is considered, its lenght is negligible. While such deviations are modelled, these deviations can be described in the form of ''impulses". Consequently, modelling impulsive problems produce differential impulsive equations in optimal control, industrial robotics, ecology, population dynamics, physics (Bainov & Simeonov, 1993; Bainov & Simeonov, 1989; Lakshmikantham et al., 1989; Samoilenko & Perestyuk, 1989) and so on.

Mahmudov in (Mahmudov, 2019) consider a delayed system having noncommutative coefficients in the classical Caputo fractional sense with the same structure as (Khusainov & Shuklin, 2003) and offer an explicit solution by proposing the delay perturbation of the two-parameter Mittag-Leffler functions. The reseachers in (Aydin et al., 2022) examine the system obtained by using the Caputo fractional derivative with respect to another function instead of the classical Caputo fractional derivative in the system of (Mahmudov, 2019). In the sequel, Aydin and Mahmudov in (Aydin & Mahmudov, 2022) take the same system as (Mahmudov, 2019) by adding an impulsive initial condition into consideration and prove its controllability in the iterative learning control sense. This time, we combine the system of (Aydin et al., 2022) with an impulsive initial condition. This makes the system (1) different from the existing studies in the literature. To the best of our knowledge, this system is firstly introduced and its relative controllability is investigated.

Inspired by the above-cited studies, we investigate the below semilinear impulsesive fractional delayed differential equations consisting of the traditional Caputo fractional derivative with respect to another function

$$\begin{cases} {}_{-r^+}^{\;\,C}D_{\varphi}^{\beta}\,\rho(\varsigma) = M\rho(\varsigma) + A\rho(\varsigma - r) + g\big(\varsigma, \rho(\varsigma)\big), & 0 < \varsigma \leq T, \quad r > 0, \\ \rho(\varsigma) = \vartheta(\varsigma), & -r \leq \varsigma \leq 0, \\ \rho(\varsigma_i^+) = \rho(\varsigma_i^-) + f\big(\rho(\varsigma_i)\big) & \varsigma_i \in J \end{cases} \tag{1}$$

where ${}_{-r^+}^{\;\,C}D_{\varphi}^{\beta}$ is φ-Caputo derivative of order $0 < \beta \leq 1$. Here, φ is a real valued increasing function on $\mathbb{R}$ and $\varphi'(t) \neq 0$, $t \in [-r, T]$, $M, A \in \mathbb{R}^{n\times n}$ which do not have to be commutative. Also, $g \in C\left([0,T] \times \mathbb{R}^n, \mathbb{R}^n\right)$, $f \in C(\mathbb{R}^n, \mathbb{R}^n)$, and $\vartheta(\varsigma) \in C^1\left([-r, 0], \mathbb{R}^n\right)$, $J = \{\varsigma_1, \varsigma_2, \dots, \varsigma_m\}$ is the impulsive times with $0 < \varsigma_1 < \cdots < \varsigma_m < T$, $T = lr$ for a fixed $l \in \mathbb{N}$. The jumps

$$\rho(\varsigma_i^+) = \lim_{\varepsilon \to 0^+} \rho(\varsigma_i + \varepsilon), \qquad \rho(\varsigma_i^-) = \lim_{\varepsilon \to 0^-} \rho(\varsigma_i + \varepsilon)$$

express the right limit and the left limit of $\rho(\varsigma)$ at $\varsigma = \varsigma_i$, each to each.

PRELIMINARIES

In this section we will present most essential tools to be used in the following sections. $\mathbb{R}^n$ is the famous Euclidean space with dimension $n \in \{1,2,3,\dots\}$. For $a, b \in \mathbb{R}$ with $a < b$, let

$$C([a,b], \mathbb{R}^n) = \{f : [a,b] \to \mathbb{R}^n : f \text{ is continuous}\}$$

with the maximum norm $\|.\|_C$, which is

$$\|f\|_C = max\{\|f(\varsigma)\|, \varsigma \in [a,b]\}$$

for $\|.\|$ is a norm on $\mathbb{R}^n$. Let $AC[a,b]$ symbolise the absolutely continuous functions' space. For $n \in \{1,2,3,\ldots\}$, $AC^n[a,b]$ the space of all complex-valued functions $f(\varsigma)$ such that $f^{(n-1)}(\varsigma) \in AC[a,b]$.

Lemma 1. (Lemma 3.4., Aydin et al., 2022) $\mathcal{X}^{M,A,\varphi}_{\beta,1,r}(\varsigma,t)$ is a solution of ${}_{-r^+}^{\;C}D^{\beta}_{\varphi}\,\rho(\varsigma) = M\rho(\varsigma) + A\rho(\varsigma - r)$, that is,

$$ {}_{-r^+}^{\;C}D^{\beta}_{\varphi}\mathcal{X}^{M,A,\varphi}_{\beta,1,r}(\varsigma,t) = M\mathcal{X}^{M,A,\varphi}_{\beta,1,r}(\varsigma,t) + A\mathcal{X}^{M,A,\varphi}_{\beta,1,r}(\varsigma,t+r). $$

Lemma 2. (Corollary 3.8., Aydin et al., 2022) A continuous solution w of the equation (1) without the impulsive initial condition is

$$ \rho(\varsigma) = \mathcal{X}^{M,A,\varphi}_{\beta,1,r}(\varsigma,-r)\vartheta(-r) + \int_{-r}^{0} \mathcal{X}^{M,A,\varphi}_{\beta,\beta,r}(\varsigma,t)\left[\left({}_{-r^+}^{\;C}D^{\beta}_{\varphi}\vartheta\right)(t) - A\vartheta(t)\right] d\varphi(t) $$
$$ + \int_{0}^{\varsigma} \mathcal{X}^{M,A,\varphi}_{\beta,\beta,r}(\varsigma,t) g\big(t,\rho(t)\big) d\varphi(t), $$

here, φ-delay perturbation of two-parameter Mittag-Leffler function $\mathcal{X}^{M,A,\varphi}_{\beta,\alpha,r}$ defined by

$$ \mathcal{X}^{M,A,\varphi}_{\beta,\alpha,r}(\varsigma,s) = \begin{cases} \Theta, & \varsigma - s \in [-r,0), \\ I, & \varsigma = s, \\ \sum_{k=0}^{\infty}\sum_{j=0}^{l-1} Q_{k+1}(jr)\frac{[\varphi(\varsigma)-\varphi(s+jr)]^{k\beta+\alpha-1}}{\Gamma(k\beta+\alpha)}, & \varsigma - s \in ((l-1)r, lr], \end{cases} \tag{2} $$

where φ is an increasing real-valued function on $\mathbb{R}$ such that $\varphi'(\varsigma) \neq 0$, $t \in [-r,T]$, I and Θ are the representations of the identity and zero matrices each to each. In the light of (Mahmudov, 2019), the recursive matrices $Q_k(s)$ are defined for $s = kr$, $k = 0,1,2,\ldots$, as

$$ Q_0(s) = \Theta, \quad Q_1(0) = \mathrm{I}, \quad Q_k(-r) = \Theta, \quad Q_{k+1}(s) = \mathrm{M}Q_k(s) + \mathrm{A}Q_k(s-r) \;. $$

Lemma 3. (Lemma 3.10., Aydin et al., 2022) If $t \in [0,T]$, $T = lr$ where $l \in \mathbb{N}$ and $r \in \mathbb{R}^+$, then the following inequality holds true:

$$ \int_0^{\varsigma} \left\|\mathcal{X}^{M,A,\varphi}_{\beta,\alpha,r}(\varsigma,s)\right\| d\varphi(s) \leq [\varphi(T) - \varphi(0)]\,\mathcal{X}^{\|M\|,\|A\|,\varphi}_{\beta,\alpha,r}(T,0)\;. $$

Lemma 4. (Lemma 3.3., Aydin et al., 2022) $\mathcal{X}^{M,A,\varphi}_{\beta,\alpha,r}(\varsigma,s)$ is a jointly continuous matrix operator in $0 < s < \varsigma < \infty$.

From here on, we will offer our fundamental contributions.

THE REPRESENTATION OF A SOLUTION

Theorem 5. A continuous solution ρ of the equation (1) is

$$ \rho(\varsigma) = \mathcal{X}^{M,A,\varphi}_{\beta,1,r}(\varsigma,-r)\vartheta(-r) + \int_{-r}^{0} \mathcal{X}^{M,A,\varphi}_{\beta,\beta,r}(\varsigma,t)\left[\left({}_{-r^+}^{\;C}D^{\beta}_{\varphi}\vartheta\right)(t) - A\vartheta(t)\right] d\varphi(t) $$
$$ + \int_{0}^{\varsigma} \mathcal{X}^{M,A,\varphi}_{\beta,\beta,r}(\varsigma,t) g(t,\rho(t)) d\varphi(t) + \sum_{0<x_i<x} \mathcal{X}^{M,A,\varphi}_{\beta,1,r}(\varsigma,\varsigma_i) f\big(\rho(\varsigma_i)\big), \quad x > 0. $$

where $\mathcal{X}^{M,A,\varphi}_{\beta,\alpha,r}$ is φ-delay perturbation of two-parameter Mittag-Leffler function given above.

Proof. If one combines Lemma 1 with Lemma 2, the proof is completed out of the satisfaction of the impulsive initial condition. Now, we will show that the solution satisfies the impulsive initial condition. For each $\varsigma \in (\varsigma_{k-1}, \varsigma_k]$ the solution $\rho(\varsigma)$ is given by

$$\rho(\varsigma) = \mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,-r)\vartheta(-r) + \int_{-r}^{0} \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,t)\left[\left({}_{-r^+}^{\;C}D_{\varphi}^{\beta}\vartheta\right)(t) - A\vartheta(t)\right]d\varphi(t)$$
$$+\int_{0}^{\varsigma} \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,t)g(t,\rho(t))d\varphi(t) + \sum_{i=0}^{k-1} \mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,\varsigma_i)\, f\big(\rho(\varsigma_i)\big),$$

and for each $\varsigma \in (\varsigma_k, \varsigma_{k+1}]$, we have

$$\rho(\varsigma) = \mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,-r)\vartheta(-r) + \int_{-r}^{0} \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,t)\left[\left({}_{-r^+}^{\;C}D_{\varphi}^{\beta}\vartheta\right)(t) - A\vartheta(t)\right]d\varphi(t)$$
$$+\int_{0}^{\varsigma} \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,t)g(t,\rho(t))d\varphi(t) + \sum_{i=0}^{k} \mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,\varsigma_i)\, f\big(\rho(\varsigma_i)\big).$$

Since it is known that $\mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma_k,\varsigma_k) = I$, we acquire

$$\rho(\varsigma_i^+) = \mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,-r)\vartheta(-r) + \int_{-r}^{0} \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,t)\left[\left({}_{-r^+}^{\;C}D_{\varphi}^{\beta}\vartheta\right)(t) - A\vartheta(t)\right]d\varphi(t)$$
$$+\int_{0}^{\varsigma} \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,t)g(t,\rho(t))d\varphi(t) + \sum_{i=0}^{k} \mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,\varsigma_i)\, f\big(\rho(\varsigma_i)\big)$$
$$= \rho(\varsigma_i^-) + \mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma_k,\varsigma_k)\, f\big(\rho(\varsigma_k)\big) = \rho(\varsigma_i^-) + f\big(\rho(\varsigma_k)\big)$$

which completes the proof.

EXISTENCE UNIQUENESS RESULTS

Unfortunately, the conditions given in the statements of the problem root is not enough to assure that the solution given in Theorem 5 is unique. So, we need to make a couple of extra assumptions as follows:

A_1 :: The function g satisfies the Lipschitz condition with $L_g > 0$,

$$\|g(\varsigma,\rho) - g(\varsigma,v)\| \le L_g\|\rho - v\|, \qquad \varsigma \in [0,T], \quad \rho, v \in \mathbb{R}^n.$$

A_2 :: The function f satisfies the Lipschitz condition with $L_f > 0$.

A_3 :: $\big([\varphi(T) - \varphi(0)]L_g + mL_f\big) max\left\{\mathcal{X}_{\beta,1,r}^{\|M\|,\|A\|,\varphi}(T,0), \mathcal{X}_{\beta,\beta,r}^{\|M\|,\|A\|,\varphi}(T,0)\right\} < 1.$

Theorem 6. Under all assumptions A_1, A_2, A_3, the integral equation given in Theorem 5 has a unique solution on $[-r,T]$.

Proof. Define $G: C\,([-r,T],\mathbb{R}^n) \to C\,([-r,T],\mathbb{R}^n)$ by

$$G\rho(\varsigma) = \mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,-r)\vartheta(-r) + \int_{-r}^{0} \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,t)\left[\left({}_{-r^+}^{\;C}D_{\varphi}^{\beta}\vartheta\right)(t) - A\vartheta(t)\right]d\varphi(t)$$
$$+ \int_{0}^{\varsigma} \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,t)g(t,\rho(t))d\varphi(t) + \sum_{0<\varsigma_i<\varsigma} \mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,\varsigma_i)\, f\big(\rho(\varsigma_i)\big).$$

By taking arbitrary $\rho, v \in C\,([-r,T],\mathbb{R}^n)$, one can obtain the following estimation of $\|G\rho(\varsigma) - Gv(\varsigma)\|$:

$$\|G\rho(\varsigma) - Gv(\varsigma)\| \le L_g \int_0^{\varsigma} \left\|\mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,s)\right\| d\varphi(s)\ \|\rho - v\|_C + L_f \sum_{0<t_i<t} \left\|\mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,\varsigma_i)\right\|\ \|\rho - v\|_C$$
$$\le \left([\varphi(T)-\varphi(0)]L_g + mL_f\right) max\left\{\mathcal{X}_{\beta,1,r}^{\|M\|,\|A\|,\varphi}(T,0), \mathcal{X}_{\beta,\beta,r}^{\|M\|,\|A\|,\varphi}(T,0)\right\}\|\rho - v\|_C$$

In the light of A_3, G is a contraction. Consequently, G owns a unique fixed point due to Banach fixed point theorem.

STABILITY RESULTS

In this section, we firstly share fundamental definition and remark to demonstrate that the equation (1) is Ulam-Hyers (UH) stable.

Definition 7. If $\forall \epsilon > 0$ and for any solution $\rho \in C\,([0,T],\mathbb{R}^n)$ of inequality

$$\left\| {}_{-r^+}^{\;C}D_{\varphi}^{\beta}\,\rho(\varsigma) - M\rho(\varsigma) - A\rho(\varsigma - r) - g\big(\varsigma,\rho(\varsigma)\big)\right\| < \epsilon \tag{3}$$

then there is a solution $v \in C\,([0,T],\mathbb{R}^n)$ of (1), and a $\sigma > 0$ such that

$$\|\rho(\varsigma) - v(\varsigma)\| < \sigma\epsilon, \quad \varsigma \in [0,T]. \tag{4}$$

Then, equation (1) is UH-stable.

Remark 8. A function $\rho \in C'\,([0,T],\mathbb{R}^n)$ is a solution of (3) iff there is at least one element $h \in C\,([0,T],\mathbb{R}^n)$ fulfilling

- $\|h(\varsigma)\| \le \epsilon$
- ${}_{-r^+}^{\;C}D_{\varphi}^{\beta}\rho(\varsigma) = M\rho(\varsigma) + A\rho(\varsigma - r) + g\big(\varsigma,\rho(\varsigma)\big) + h(\varsigma).$

Theorem 9. Under all of circumstances in Theorem 6, the system (1) is stable in the sense of Ulam-Hyers.

Proof. Suppose $\rho \in C\,([0,T],\mathbb{R}^n)$ that fulfils (3), and let $v \in C\,([0,T],\mathbb{R}^n)$ which is the unique solution of system (1) with the initial condition $v(\varsigma) = \rho(\varsigma)$ for all $\varsigma \in [-r,0]$, $\rho(\varsigma_i^+) - \rho(\varsigma_i^-) = v(\varsigma_i^+) - v(\varsigma_i^-) = f\big(\rho(\varsigma_i)\big)$. Based on Remark 8 and the rule of G, one acquires

$$\|h(\varsigma)\| \le \epsilon, \quad \rho(\varsigma) = G\rho(\varsigma) + \int_0^{\varsigma} \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,t)h(t)d\varphi(t),$$

and also $v(\varsigma) = Gv(\varsigma)$ for each $\varsigma \in [0,T]$. One gets

$$\|G\rho(\varsigma) - \rho(\varsigma)\| \le \int_0^{\varsigma} \left\|\mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,s)\right\| \|h(s)\|d\varphi(s) \le [\varphi(T)-\varphi(0)]\mathcal{X}_{\beta,\beta,r}^{\|M\|,\|A\|,\varphi}(T,0)\epsilon.$$

We are set to make an estimation $\|v(\varsigma) - \rho(\varsigma)\|$:

$$\|v(\varsigma) - \rho(\varsigma)\| \le \|v(\varsigma) - G\rho(\varsigma)\| + \|G\rho(\varsigma) - \rho(\varsigma)\|$$

$$\leq \left([\varphi(T)-\varphi(0)]L_g + mL_f\right) max\left\{\mathcal{X}_{\beta,1,r}^{\|M\|,\|A\|,\varphi}(T,0), \mathcal{X}_{\beta,\beta,r}^{\|M\|,\|A\|,\varphi}(T,0)\right\} \|\rho - v\|_C$$
$$+[\varphi(T)-\varphi(0)]\mathcal{X}_{\beta,\beta,r}^{\|M\|,\|A\|,\varphi}(T,0)\epsilon$$

which provides

$$\|v-\rho\|_C \leq \sigma\epsilon,$$

where

$$\sigma = \frac{[\varphi(T)-\varphi(0)]\mathcal{X}_{\beta,\beta,r}^{\|M\|,\|A\|,\varphi}(T,0)}{1-\left([\varphi(T)-\varphi(0)]L_g + mL_f\right) max\left\{\mathcal{X}_{\beta,1,r}^{\|M\|,\|A\|,\varphi}(T,0), \mathcal{X}_{\beta,\beta,r}^{\|M\|,\|A\|,\varphi}(T,0)\right\}} > 0.$$

This last point completes the proof.

RELATIVE CONTROLLABILITY RESULTS

In current section we relatively investigate the controllability of the impulsive fractional delayed differential systems having Caputo fractional derivatives w.r.t. another function while it is not only linear but also semilinear.

Definition 10. System (1) is called relatively controllable, if there is a control $u \in L^2(I=[0,T],\mathbb{R}^n)$ so that equation (1) owns a solution $\rho \in C([-r,\tau],\mathbb{R}^n)$ that holds the initial delayed condition, the initial impulsive condition, and $\rho(\tau) = \rho_\tau$ for the arbitrary final value $\rho_\tau \in \mathbb{R}^n$ with the arbitrary time τ, any initial continuously differentiable $\mathbb{R}^n$-valued function φ on $[-\mathrm{r},0]$.

There are two cases for the system (1) to investigate its relative controllability. If the system (1) is without the semilinear term $g(\varsigma,\rho(\varsigma))$, $0<\varsigma\leq T$, it is called the linear case of the system (1). Otherwise, it is called the semilinear case of the system (1). We will consider these two cases individually as follows.

The Relative Controllability of the Linear Case of the System (1).

We will consider the following control system

$$\begin{cases} {}_{-r^+}^{C}D_{\varphi}^{\beta}\,\rho(\varsigma) = M\rho(\varsigma) + A\rho(\varsigma-r) + Su(\varsigma), & 0<\varsigma\leq T, \quad r>0, \\ \rho(\varsigma) = \vartheta(\varsigma), & -r\leq\varsigma\leq 0, \\ \rho(\varsigma_i^+) = \rho(\varsigma_i^-) + f(\rho(\varsigma_i)) & \varsigma_i \in J \end{cases} \tag{5}$$

whose solution is given by

$$\rho(\varsigma) = \mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,-r)\vartheta(-r) + \int_{-r}^{0} \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,s)\left[\left({}_{-r^+}^{C}D_{\varphi}^{\beta}\vartheta\right)(s) - A\vartheta(s)\right]d\varphi(s)$$
$$+\int_{0}^{\varsigma} \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,s)Su(s)d\varphi(s) + \sum_{0<\varsigma_i<\varsigma} \mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,\varsigma_i)f(\rho(\varsigma_i)),$$

here $u \in \mathbb{R}^n$ is a control function and $S \in \mathbb{R}^{n\times n}$.

Theorem 11. The system (5) is relatively controllable if and only if the following Gramian matrix

$$W[0,\tau] = \int_{0}^{\tau} \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\tau,s)SS^{*}\mathcal{X}_{\beta,\beta,r}^{M^*,A^*,\varphi}(\tau,s)d\varphi(s)$$

is nonsingular, where $.^*$ stands for the transpose of a matrix.

Proof. ⇒: Let $W[0,\tau]$ be singular while the system (5) is relatively controllable. There is **a** nonzero $\pi \in \mathbb{R}^n$,

$$W[0,\tau]\,\pi = 0.$$

One gets

$$\int_0^\tau \pi^* \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\tau,s) S S^* \mathcal{X}_{\beta,\beta,r}^{M^*,A^*,\varphi}(\tau,s)\pi d\varphi(s) = 0,$$

which provides

$$\pi^* \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\tau,s) S = 0, \quad 0 \le s \le \tau.$$

Based on the relative controllability of the system, we can find u_1 and u_2 for the different final $0, \pi \in \mathbb{R}^n$,respectively so that

$$\pi^*\pi = \int_0^\tau \pi^* \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\tau,s) S\left(u_2(s) - u_1(s)\right) d\varphi(s) = 0$$

from which $\pi = 0$ is obtained. This is a contradiction.

⇐: By means of the invertibility of the Gramian matrix, it is known that its inverse $W^{-1}[0,\tau]$ exists. If one regards the following continuous function

$$u(\varsigma) = S^* \mathcal{X}_{\beta,\beta,r}^{M^*,A^*,\varphi}(\tau,\varsigma)\, W^{-1}[0,\tau]\ \vartheta$$

where

$$\vartheta(\varsigma) = \rho_\tau - \mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,-r)\vartheta(-r) - \int_{-r}^{0} \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,t)\left[\left({}_{-r^+}^{\ C}D_\varphi^\beta \vartheta\right)(t) - A\vartheta(t)\right] d\varphi(t)$$
$$- \sum_{0<\varsigma_i<\varsigma} \mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,\varsigma_i)\, f\left(\rho(\varsigma_i)\right).$$

as a control, one can easily observe $\rho(\tau) = \rho_\tau$, and w fulfills all of the initial conditions

The Relative Controllability of the Semilinear Case of the System (1).

We will consider the following control system

$$\begin{cases} {}_{-r^+}^{\ C}D_\varphi^\beta\, \rho(\varsigma) = M\rho(\varsigma) + A\rho(\varsigma - r) + g\left(\varsigma, \rho(\varsigma)\right) + Su(\varsigma), & 0 < \varsigma \le T, \quad r > 0, \\ \rho(\varsigma) = \vartheta(\varsigma), & -r \le \varsigma \le 0, \\ \rho(\varsigma_i^+) = \rho(\varsigma_i^-) + f\left(\rho(\varsigma_i)\right) & \varsigma_i \in J \end{cases} \tag{6}$$

whose solution is given by

$$\rho(\varsigma) = \mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,-r)\vartheta(-r) + \int_{-r}^{0} \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,t)\left[\left({}_{-r^+}^{\ C}D_\varphi^\beta \vartheta\right)(t) - A\vartheta(t)\right] d\varphi(t)$$

$$+\int_0^x \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,t)g\big(t,\rho(t)\big)d\varphi(t)+\sum_{0<\varsigma_i<\varsigma}\mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,\varsigma_i)\,f\big(\rho(\varsigma_i)\big).$$
$$+\int_0^\varsigma \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,t)Su(t)d\varphi(t),\quad \varsigma>0.$$

Unfortunately, we can not control this system without putting extra conditions on the nonlinear function and impulsive function, an extra operator. Now, let us make some assumptions as follows:

A_4 :: The operator $M: L^2(I,\mathbb{R}^n)\longrightarrow \mathbb{R}^n$

$$Mu=\int_0^\tau \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\tau,s)Su(s)d\varphi(s),$$

owns an inverse M^{-1} which taking values in $L^2\big(I,\mathbb{R}^n\big)/kerM$. Let X_i, $i=1,2$, be Banach spaces. $B(X_1,X_2)$ consisting of all both bounded and linear is endowed with the norm $\|.\|_B$. For simplicity, we will set

$$R\coloneqq \|M^{-1}\|_{B\left(\mathbb{R}^n,L^2(I,\mathbb{R}^n)/kerM\right)},$$
$$R_1\coloneqq \left\|\mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,-r)\vartheta(-r)\right\|+\left\|\int_{-r}^0 \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,s)\left[\left({}_{-r^+}^{\;C}D_\varphi^\beta\vartheta\right)(s)-A\vartheta(s)\right]d\varphi(s)\right\|,$$
$$R_2\coloneqq \sum_{0<\varsigma_i<\varsigma}\left\|\mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,\varsigma_i)\right\||f(0)|+[\varphi(T)-\varphi(0)]\mathcal{X}_{\beta,\beta,r}^{\|M\|,\|A\|,\varphi}(T,0)\max_{[0,T]}|g(\varsigma,0)|,$$
$$R_3\coloneqq \left(L_f\sum_{0<\varsigma_i<\varsigma}\left\|\mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,\varsigma_i)\right\||f(0)|+L_g[\varphi(T)-\varphi(0)]\mathcal{X}_{\beta,\beta,r}^{\|M\|,\|A\|,\varphi}(T,0)\right)\|\rho\|_C .$$

From Remark 3.3. of (Wang et al., 2017),

$$R=\sqrt{\|W^{-1}[0,\tau]\|}.$$

Theorem 12. Suppose that $1\geq\beta>0.5$. Under the assumptions A_1, A_2, and A_4 are fulfilled. Then the system (6) is relatively controllable if

$$(1+R\|S\|max\{1,R_3\})R_3<1.$$

Proof. Based on the assumption A_4, one can define the following control function

$$u_\rho=M^{-1}\left[\rho_\tau-\mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,-r)\vartheta(-r)-\int_{-r}^0\mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,s)\left[\left({}_{-r^+}^{\;C}D_\varphi^\beta\vartheta\right)(s)-A\vartheta(s)\right]d\varphi(s)\right]$$
$$+M^{-1}\left[-\sum_{0<\varsigma_i<\varsigma}\mathcal{X}_{\beta,1,r}^{M,A,\varphi}(\varsigma,\varsigma_i)\,f\big(\rho(\varsigma_i)\big)-\int_0^\varsigma\mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,s)g\big(s,\rho(s)\big)d\varphi(s)\right].$$

By executing this control function, one can also define $K: C\,(I,\mathbb{R}^n)\rightarrow C\,(I,\mathbb{R}^n)$ by

$$K\rho(\varsigma) = \mathcal{X}^{M,A,\varphi}_{\beta,1,r}(\varsigma,-r)\vartheta(-r) + \int_{-r}^{0} \mathcal{X}^{M,A,\varphi}_{\beta,\beta,r}(\varsigma,t)\left[\left({}_{-r^+}^{C}D^{\beta}_{\varphi}\vartheta\right)(t) - A\vartheta(t)\right]d\varphi(t)$$
$$+\int_{0}^{\varsigma} \mathcal{X}^{M,A,\varphi}_{\beta,\beta,r}(\varsigma,t)g\big(t,\rho(t)\big)d\varphi(t) + \sum_{0<\varsigma_i<\varsigma} \mathcal{X}^{M,A,\varphi}_{\beta,1,r}(\varsigma,\varsigma_i)\,f\big(\rho(\varsigma_i)\big).$$
$$+\int_{0}^{\varsigma} \mathcal{X}^{M,A,\varphi}_{\beta,\beta,r}(\varsigma,t)Su_{\rho}(t)d\varphi(t),\ \ \varsigma > 0.$$

Now, we need to determine such a radius r for $D_r \coloneqq \{\rho \in C\,(I,\mathbb{R}^n) :\ \|\rho\|_C \le r\}$ which is a convex, closed and bounded subset that $K(D_r)\ \subseteq\ D_r$. To do this, start with the norm of the control function:

$$\left\|u_{\rho}\right\| \le R(R_1 + R_2 + R_3\|\rho\|_C).$$

The norm of the operator $K\rho(\varsigma)$ for $\rho \in D_r$ is

$$\|K\rho(\varsigma)\,\| \le R_1 + R_2 + R_3\|\rho\|_C + R\|S\|(R_1 + R_2 + R_3\|\rho\|_C).$$

If we take

$$r = \frac{(1 + R\|S\|)(R_1 + R_2) + R\|S\|\|w_{\tau}\|}{1 - (1 + R\|S\|max\{1,R_3\})R_3} > 0,$$

the desired thing is demonstrated. Now we will separate K in two different operators as follows:

$$K_1\rho(\varsigma) = \mathcal{X}^{M,A,\varphi}_{\beta,1,r}(\varsigma,-r)\vartheta(-r) + \int_{-r}^{0} \mathcal{X}^{M,A,\varphi}_{\beta,\beta,r}(\varsigma,s)\left[\left({}_{-r^+}^{C}D^{\beta}_{\varphi}\vartheta\right)(s) - A\vartheta(s)\right]d\varphi(s)$$
$$+\int_{0}^{\varsigma} \mathcal{X}^{M,A,\varphi}_{\beta,\beta,r}(\varsigma,s)Su_{\rho}(s)d\varphi(s) + \sum_{0<\varsigma_i<\varsigma} \mathcal{X}^{M,A,\varphi}_{\beta,1,r}(\varsigma,\varsigma_i)\,f\big(\rho(\varsigma_i)\big), \qquad \varsigma \in I,$$

and

$$K_2w(\varsigma) = \int_{0}^{\varsigma} \mathcal{X}^{M,A,\varphi}_{\beta,\beta,r}(\varsigma,s)g\big(s,\rho(s)\big)d\varphi(s),\ \ \varsigma \in I.$$

For $\rho, v \in D_r$, one gets

$$\left\|u_{\rho}(\varsigma) - u_v(\varsigma)\right\| \le RR_3\|\rho(\varsigma) - v(\varsigma)\|$$

and

$$\begin{aligned}\|K_1\rho(\varsigma) - K_1v(\varsigma)\| &\le [\varphi(T) - \varphi(0)]\mathcal{X}^{\|M\|,\|A\|,\varphi}_{\beta,\beta,r}(T,0)\|S\|\,\left\|u_{\rho}(\varsigma) - u_v(\varsigma)\right\| \\ &\quad + L_f \sum_{0<\varsigma_i<\varsigma} \left\|\mathcal{X}^{M,A,\varphi}_{\beta,1,r}(\varsigma,\varsigma_i)\right\|\,\|\rho - v\|_C \\ &\le [\varphi(T) - \varphi(0)]\mathcal{X}^{\|M\|,\|A\|,\varphi}_{\beta,\beta,r}(T,0)\|S\|RR_3\|\rho - v\|_C \\ &\quad + L_f \sum_{0<\varsigma_i<\varsigma} \left\|\mathcal{X}^{M,A,\varphi}_{\beta,1,r}(\varsigma,\varsigma_i)\right\|\,\|\rho - v\|_C \\ &\le (1 + R\|S\|max\{1,R_3\})R_3\|\rho - v\|_C,\end{aligned}$$

which gives that K_1 is a contraction. Assume that $\rho_n \in D_r$ with $\rho_n \longrightarrow \rho$ in D_r . Since g is continuous, $g(\varsigma,\rho_n(\varsigma)) \rightarrow g(\varsigma,\rho(\varsigma))$. By using dominated convergence theorem

$$\|K_2\rho_n(\varsigma) - K_2\rho(\varsigma)\| \leq \int_0^{\varsigma} \left\|\mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,s)\right\| \left\|g\big(s,\rho_n(s)\big) - g\big(s,\rho(s)\big)\right\| d\varphi(s),$$

goes to zero as n tends to ∞. Thus, K_2 is continuous on D_r. The last task is to show that K_2 is compact. For $\rho \in D_r$, $0 < \varsigma < \varsigma + h < \tau$

$$K_2\rho(\varsigma+h) - K_2\rho(\varsigma) = \int_t^{\varsigma+h} \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma+h,s) g\big(s,\rho(s)\big)d\varphi(s)$$
$$+ \int_0^{\varsigma} \left(\mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma+h,s) - \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,s)\right) g\big(s,\rho(s)\big)d\varphi(s).$$

Introduce the below notations:

$$\lambda_1 := \int_{\varsigma}^{\varsigma+h} \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma+h,s) g\big(s,\rho(s)\big)d\varphi(s),$$
$$\lambda_2 := \int_0^{\varsigma} \left(\mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma+h,s) - \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,s)\right) g\big(s,\rho(s)\big)d\varphi(s)\,.$$

With an easy calculation, one can acquire

$$\|\lambda_1\| \leq \left(L_g\|\rho\|_C + \max_{[0,T]}|g(\varsigma,0)|\right) \int_{\varsigma}^{\varsigma+h} \left\|\mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma+h,s)\right\| d\varphi(s) \longrightarrow 0,$$
$$\|\lambda_2\| \leq \left(L_g\|\rho\|_C + \max_{[0,T]}|g(\varsigma,0)|\right) \int_0^{\varsigma} \left\|\mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma+h,s) - \mathcal{X}_{\beta,\beta,r}^{M,A,\varphi}(\varsigma,s)\right\| d\varphi(s) \longrightarrow 0,$$

as $h \longrightarrow 0$. As a result, one acquires

$$\|K_2\rho(\varsigma+h) - K_2\rho(\varsigma)\| \leq \|\lambda_1\| + \|\lambda_2\| \longrightarrow 0 \;\; as \;\; h \longrightarrow 0.$$

$K_2(D_r)$ is uniformly bounded because one easily reach to the following upper bound for all members of $K_2(D_r)$ with the familiar computations,

$$\|K_2\rho\| \leq \left(L_g r + \max_{[0,T]}|g(\varsigma,0)|\right)\tau\, \mathcal{X}_{\beta,\beta,r}^{\|M\|,\|A\|,\varphi}(\tau,0).$$

Because of the equicontinuity and uniform boundedness of K_2, Arzela-Ascoli theorem provides K_2 is compact. Due to the fixed-point theorem of Krasnoselskii, K owns a fixed point $\rho \in D_r$.

CONCLUSION

The current paper is, in brief, devoted to investigating the both uniqueness and existence of the solution and examining stability and controllability of the discussed equations. The obtained results are quite comprehensive and cover many studies which are not available in the literature because the Caputo fractional derivative with respect to another function reduces to the classical Caputo fractional derivative in the case of $\varphi(\varsigma) = \varsigma$ and Hadamard fractional derivate when $\varphi(\varsigma) = \ln \varsigma$. For a next problem, the following neutral fractional system:

$$\begin{cases} {}_{-r^+}^{C}D_{\varphi}^{\beta}\, \rho(\varsigma) - N\, {}_{-r^+}^{C}D_{\varphi}^{\beta}\rho(\varsigma - r) = M\rho(\varsigma) + A\rho(\varsigma - r) + g\big(\varsigma, \rho(\varsigma)\big), & 0 < \varsigma \le T, \quad r > 0, \\ \rho(\varsigma) = \vartheta(\varsigma), & -r \le \varsigma \le 0, \\ \rho(\varsigma_i^+) = \rho(\varsigma_i^-) + f\big(\rho(\varsigma_i)\big) & \varsigma_i \in J \end{cases}$$

where $N \in \mathbb{R}^n$ and the remaining information is given in (1), can be taken into consideration and the stability and controllability of this neutral fractional system can be investigated in addition to the fact that both the uniqueness and existence of its solution are examined.

REFERENCES

Aydin, M., Mahmudov, N. I., Aktuğlu, H., Baytunç, E., & Atamert, M. S. (2022). On a study of the representation of solutions of a Ψ-Caputo fractional differential equations with a single delay. Electronic Research Archive, 30, 1016–1034.

Aydin, M., & Mahmudov, N. I. (2022). Iterative learning control for impulsive fractional order time-delay systems with nonpermutable constant coefficient matrices. International Journal of Adaptive Control and Signal Processing, 36(1), 1419–1438.

Bainov, D. D., & Simeonov, P. S. (1989). Systems with Impulse Effect. Ellis Horwood Series:Mathematics and Its Applications, Ellis Horwood, Chichester.

Bainov, D. D., & Simeonov, P. S. (1993). Impulsive Differential Equations: Periodic Solutions and Applications, Pitman Monographs and Surveys in Pure and Applied Mathematics. (66 vol.). Longman Scientific & Technical, Harlow; JohnWiley & Sons, New York.

Elshenhab, A. M., & Wang, X. T. (2021a). Representation of solutions for linear fractional systems with pure delay and multiple delays. Mathematical Methods in the Applied Sciences, 44, 12835–12850.

Elshenhab, A. M., & Wang, X. T. (2021b). Representation of solutions of linear differential systems with pure delay and multiple delays with linear parts given by non-permutable matrices. *Applied Mathematics and Computation*, 410, 126443. https://doi.org/10.1016/j.amc.2021.126443

Khusainov, D. Y., & Shuklin, G. V. (2005). Relative controllability in systems with pure delay. *International Journal of Applied Mathematics*, 2, 210–221. https://doi.org/10.1007/s10778-005-0079-3

Khusainov, D. Y., & Shuklin, G. V. (2003). Linear autonomous time-delay system with permutation matrices solving. *Stud. Univ. Zilina*, 17, 101–108.

Lakshmikantham, V., Bainov, D. D., & Simeonov, P. S. (1989). Theory of Impulsive Differential Equations, Series in Modern Applied Mathematics. (6 vol.). World Scientific, New Jersy. https://doi.org/10.1142/0906

Liang, C., Wang, J., & O'Regan, D. (2017). Controllability of nonlinear delay oscillating systems. *Electronic Journal of Qualitative Theory of Differential Equations*, 2017, 1–18. https://doi.org/10.14232/ejqtde.2017.1.47

Li, M., & Wang, J. R. (2018). Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations. *Applied Mathematics and Computation*, 324, 254–265. https://doi.org/10.1016/j.amc.2017.11.063

Liu, L., Dong, Q., & Li, G. (2021). Exact solutions and Hyers–Ulam stability for fractional oscillation equations with pure delay. *Applied Mathematics Letters*, 112, 106666. https://doi.org/10.1016/j.aml.2020.106666

Mahmudov, N. I., & Aydın, M. (2021). Representation of solutions of nonhomogeneous conformable fractional delay differential equations. *Chaos Solitons Fractals*, 150, 111190. https://doi.org/10.1016/j.chaos.2021.111190

Mahmudov, N. I. (2022). Multi-delayed perturbation of Mittag-Leffler type matrix functions. *Journal of Mathematical Analysis and Applications*, 505, 125589. https://doi.org/10.1016/j.jmaa.2021.125589

Mahmudov, N. I. (2019). Delayed perturbation of Mittag-Leffler functions and their applications to fractional linear delay differential equations. *Mathematical Methods in the Applied Sciences*, 42, 5489–5497. https://doi.org/10.1002/mma.5446

Mahmudov, N. I. (2018). Representation of solutions of discrete linear delay systems with non permutable matrices. *Applied Mathematics Letters*, 85, 8–14. https://doi.org/10.1016/j.aml.2018.05.015

Samoilenko, A. M., & Perestyuk, N. A. (1995). Impulsive Differential Equations, World Scientific Serieson Nonlinear Science. Series A:Monographs and Treatises, vol. 14, World Scientific, New Jersey, ISBN: 978-981-02-2416-5. https://doi.org/10.1142/2892

Wang, J., Luo, Z., & Feckan, M. (2017). Relative controllability of semilinear delay differential systems with linear parts defined by permutable matrices. *European Journal of Control*, 38, 39–46. https://doi.org/10.1016/j.ejcon.2017.08.002

You, Z., Feckan, M., & Wang, J. (2020). Relative Controllability of Fractional Delay Differential Equations via Delayed Perturbation of Mittag-Leffler Functions. *Journal of Computational and Applied Mathematics*, 378, 112939. https://doi.org/10.1016/j.cam.2020.112939.