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Fractional Order Integration: A New Perspective based on Karcı's Fractional Order Derivative

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Abstract— There are many methods/definitions for fractional order derivatives, and naturally, there are many definitions for fractional order integrals based on these definitions. In this paper, a new definition for fractional order integral was emphasized based on the definition for fractional order derivative made by Karcı.

Keywords. Fractional Calculus, Integration, Fractional Order Derivatives.

1. Introduction

The concept of rate of change in any function versus change in the independent variables was defined as derivative, and this concept attracted by many scientists and mathematicians such as Newton, L'Hospital, Leibniz, Abel, Euler, Riemann, etc (Newton, 1687; L'Hospital, 1696; L'Hospital, 1715; Goldenbaum et al, 2008; Baron, 1969; Leibniz, 1695). After these mathematicians/scientists, there are many mathematicians dealt with this concept, and some of them such as Euler, Riemann-Liouville, Caputo, Abel, Fourier, Miller-Ross, Grunwald-Letnikov, Oldham-Spanier, and Kolwankar-Gangal extended the derivative concept to fractional order derivative.

The fractional calculus, fractional order derivatives have attracted many mathematicians / scientists such as Newton, Leibniz, L'Hospital, Abel, Euler, Rieman, Fourier, Caputo, Liouville, etc and there are a lot of studies on fractional calculus (Das, 2011; Mandelbrot et al, 1968; Mirevski et al, 2007; Schiavone et al, 1990; Bataineh et al, 2009; Diethelm et al, 2005; Li et al, 2011). Karcı and his friends defined a new concept for fractional order derivatives and determined revealed the properties of fractional order derivatives (Karcı, 2013a; Karcı 2013b; Karcı, 2015a; Karcı, 2015b; Karcı, 2015c; Karcı, 2015d; Karcı, 2015e; Karcı, 2016; Karcı, 2017).

Especially, Karcı defined fractional order derivatives with holding all properties of Newtonian derivative. Due to this case, the fractional order integration was re-defined in this paper based on KArcı's fractional order derivative.

This paper is organized as follow. Section 2 describes the new definition for fractional order derivative. Section 3 illustrates the chain rule for fractional order derivative. Finally, paper is concluded in Section 4.

2. Karcı's Fractional Order Derivative

The concept of fractional order derivative was re-defined by Karcı holding the all properties of classical derivative defined by Newton. Some properties of fractional order derivative defined by Karcı are as follow. The Karcı's derivative ${}^{\partial}_{\alpha}K$ obeys the rules and properties of classical derivative (Newtonian derivative) in case of product, quotient, chain.

Definition 1: *Assume that f(x):R→R is a function, α∈R and L(.) be a L'Hospital process. The fractional order derivative of f(x) is*

$$ {}^{\partial}_{\alpha}K\, f(x) = \lim_{h\to 0} L\left(\frac{f^{\alpha}(x+h)-f^{\alpha}(x)}{(x+h)^{\alpha}-x^{\alpha}}\right) = \lim_{h\to 0}\frac{\dfrac{d\left(f^{\alpha}(x+h)-f^{\alpha}(x)\right)}{dh}}{\dfrac{d((x+h)^{\alpha}-x^{\alpha})}{dh}} = \left(\frac{f(x)}{x}\right)^{\alpha-1}\frac{df(x)}{dx} $$

Definition 2: *Assume that f(x), g(x):R→R are continuous functions, α∈R and h(x)=f(x)g(x). The fractional order derivative of h(x) is as follows.*

$$ {}^{\partial}_{\alpha}K\, h(x) = \left(\frac{f(x)g(x)}{x}\right)^{\alpha-1}\left(\frac{df(x)}{dx}g(x)+f(x)\frac{dg(x)}{dx}\right) $$

.

Definition 3: *Assume that f(x), g(x):R→R is a continue functions, α∈R and* $h(x)=\frac{f(x)}{g(x)}$. *The fractional order derivative of h(x) is as follows.*

$$ {}^{\partial}_{\alpha}K\, h(x) = \left(\frac{f(x)}{xg(x)}\right)^{\alpha-1}\left(\frac{\dfrac{df(x)}{dx}g(x)-f(x)\dfrac{dg(x)}{dx}}{g^{2}(x)}\right) $$

Definition 4: *The chain rule for fractional order derivative with respect to Definition 1 is*

$$ \frac{\partial^{\alpha}y}{\partial x^{\alpha}} = \frac{\partial^{\alpha}y}{\partial u^{\alpha}}\frac{\partial^{\alpha}u}{\partial x^{\alpha}} = \left(\frac{f(u)}{u}\right)^{\alpha-1} f'(u)\left(\frac{g(x)}{x}\right)^{\alpha-1} g'(x) $$

3. Fractional Order Integration

The fractional order integrals defined by many researchers, and the Riemann-Liouville fractional integrals (Kilbas et al, 2006) can be given as follow.

$$ \left(I^{\alpha}_{a+}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{f(t)dt}{(x-t)^{1-\alpha}}, \quad (x>a;\ R(a)>0) \text{ and} $$

$$ \left(I^{\alpha}_{a-}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{b}\frac{f(t)dt}{(t-x)^{1-\alpha}}, \quad (x<b;\ R(a)>0) $$

where Γ(α) is the gamma function. In this paper, fractional order integral is defined based on ${}^{\partial}_{\alpha}K$ fractional order derivative. The following theorem gives the fractional order integral based on ${}^{\partial}_{\alpha}K$.

Theorem: *Assume that f(x) is real function which has single polynomial term, α∈R (α is the fractional order of derivative) and* ${}^{\partial}_{\alpha}Kf(x) = bx^{c}$ *then* ${}^{i}_{\alpha}Kf(x) = \sqrt[\alpha]{\frac{\alpha b}{c+\alpha}}\,x^{\frac{c}{\alpha}+1}$.

Proof: Assume that f(x) is a real function and has single polynomial term such as $f(x)=ax^n$, a,n$\in$R, and

$$ {}_{\alpha}^{\partial}Kf(x)=\left(\frac{f(x)}{x}\right)^{\alpha-1}\frac{df(x)}{dx}=\left(\frac{ax^n}{x}\right)^{\alpha-1}anx^{n-1}=a^{\alpha-1}x^{(n-1)(\alpha-1)}anx^{\alpha-1}=a^{\alpha}nx^{\alpha(n-1)} $$

.

If ${}_{\alpha}^{\partial}Kf(x)=bx^c$, then b and c can be obtained as follows.

$$ c=\alpha(n-1)\Rightarrow n=\frac{c}{\alpha}+1 \text{ and } b=a^{\alpha}n=a^{\alpha}\left(\frac{c+\alpha}{\alpha}\right)\Rightarrow a^{\alpha}=\frac{\alpha b}{c+\alpha}\Rightarrow a=\sqrt[\alpha]{\frac{\alpha b}{c+\alpha}} \blacksquare $$

Example 1: Assume that f(x)=4x^5, and $\alpha=\frac{1}{2}$ then ${}_{\frac{1}{2}}^{\partial}Kf(x)=10x^2$. The fractional order integral of ${}_{\frac{1}{2}}^{\partial}Kf(x)=10x^2$ can be obtained as follow.

$n=\frac{2}{\frac{1}{2}}+1=5$ and $a=\sqrt[1/2]{\frac{\frac{10}{2}}{2+\frac{1}{2}}}=\sqrt[1/2]{2}=4$. So, $f(x)={}_{\frac{1}{2}}^{i}Kf(x)=ax^n=4x^5$, and this is the original function f(x).

Example 2: Assume that f(x)=4x^5, and $\alpha=2$ then ${}_{2}^{\partial}Kf(x)=80x^8$. The fractional order integral of ${}_{2}^{\partial}Kf(x)=80x^8$ can be obtained as follow.

$n=\frac{8}{2}+1=5$ and $a=\sqrt{\frac{2x80}{8+2}}=4$. So, $f(x)={}_{2}^{i}Kf(x)=ax^n=4x^5$, and this is the original function f(x).

Example 3: Assume that f(x)=4x^5, and $\alpha=-1$then ${}_{-1}^{\partial}Kf(x)=\frac{5}{4}x^{-4}$. The fractional order integral of ${}_{-1}^{\partial}Kf(x)=80x^{-6}$ can be obtained as follow.

$n=\frac{-4}{-1}+1=5$ and $a=\sqrt[-1]{\frac{-\frac{5}{4}}{-4-1}}=\left(\frac{1}{4}\right)^{-1}=4$. So, $f(x)={}_{-1}^{i}Kf(x)=ax^n=4x^5$, and this is the original function f(x).

4. Conclusions

In this paper, the methods for fractional order integral was made based on the Karcı's fractional order derivative providing the all properties of Newtonian derivative. The new definition for integral is for real function not complex functions, this definition was supported with examples.

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