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Exponential Type Estimators Using Sub-Sampling Method with Applications in Agriculture

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ABSTRACT

In this article, the family of exponential type estimators with the auxiliary variable is proposed in the case of non-response scheme for the purpose of obtaining the unknown population mean of the study variable. The non-response scheme is examined under two main cases as Case I and Case II. The bias, mean square error (MSE) and minimum MSE of the proposed family of estimators are obtained in detail for both cases. After theoretical inferences, empirical studies are carried out to show the appropriateness of the proposed family of estimators in the field of agriculture. The MSE and PRE (Percentage Relative Efficiency) values are obtained. According to the results, the proposed estimators provide more efficient results than existing estimators in the literature under the obtained conditions for both cases. We conclude that the proposed family of estimators can be applied to the agriculture data successfully.

Keywords: Agriculture, Efficiency, Non-response, Population mean estimator, Simple random sampling

1. Introduction

Instead of population, working on a sample plays an important role in terms of time, money, and labor force. The parameters of the population, such as total, mean, proportions and variance can be obtained via estimators in sample surveys. Here, the main aim is to propose more efficient estimator than other estimators in literature. For this reason, the information of the auxiliary variable is used extensively in order to obtain the efficient results. In literature, we may see various types of estimators, such as ratio, regression, product, exponential and so on in the presence of the auxiliary variable. In these types of estimators, if the relation between study (y) and auxiliary variable (x) is a straight line passing through the origin, the usual regression, ratio and product types of estimators have equal efficiencies. However, when the line does not pass through the origin, the regression, ratio and product types of estimators do not have equal efficiencies (Solanki et al. 2012). For this reason, exponential type of estimators becomes prominent among others.

In general, the estimators are proposed when all information on variables is present. However, some units on several variables cannot be obtained all the time. This situation is the most important problem in the sample survey that is named as non-response (Dansawad 2019). In order to deal with this situation, Hansen and Hurwitz (1946) introduced a new technique of sub-sampling. Here, the main aim is to reduce the effect of non-response using both response and non-response units in the estimator.

In this technique, from a population size of N units, as $S=(S_1,S_2,...,S_N)$, a sample of size n units is drawn. However, the population size N $(N_1+N_2=N)$ is divided into two groups, N_1 and N_2, as respondent and non-respondent units, respectively. Similarly, the response units are available only on n_1 while n_2 $(n_2=n-n_1)$ units are obtained as non-response. Due to the extra effort, a sub-sample size of $r=\frac{n_2}{z}$ $(z>1)$ is drawn from n_2. Thus, the population mean can be estimated by using (n_1+r) units, instead of n units, in this sub-sampling technique. Note that z is the inverse of the sampling rate whose different values are used for the MSE and PRE values.

The unbiased estimator is defined by Hansen and Hurwitz (1946) to estimate the population mean for the first time in the sub-sampling technique as follows:

$$t_H = w_1\bar{y}_1 + w_2\bar{y}_{2(r)}, \tag{1}$$

whose variance is given by

$$V(t_H) = \bar{Y}^2\left(\lambda C_y^2 + \frac{W_2(z-1)}{n}C_{y(2)}^2\right). \tag{2}$$

In Equation (1), $w_1 = \frac{n_1}{n}$ and $w_2 = \frac{n_2}{n}$ refer the weights of response and non-response units for the sample, respectively. Besides, $\bar{y}_1$ and $\bar{y}_{2(r)}$ denote the sample means of y based on n_1 and r $\left(r = {n_2}/{z}\right)$ units. In Equation (2), $f = \frac{n}{N}$, $\lambda = \frac{1-f}{n}$, $C_y^2 = \frac{S_y^2}{\bar{Y}^2}$ and $C_{y(2)}^2 = {S_{y(2)}^2}/{\bar{Y}^2}$. In addition, $W_2 = {N_2}/{N}$ is the weight of N_2 units in the population.

The non-response situation is examined under Case I and Case II, separately. In Case I, units of non-response exist only on *y*. In Case II, units of non-response exist on both *y* and *x*. In addition to this, the population mean of *x* is known for both cases.

After the sub-sampling method and pioneer study of Hansen and Hurwitz (1946), many estimators have been proposed in the literature considering the non-response scheme. Some of the estimators for the population mean are given in Tables 1 and 2, according to the Cases I and II, respectively.

Using the sub-sampling method, Rao (1986) proposed the classical ratio (t_{R1}) and regression (t_{reg1}) estimators under the Case I. Singh et al. (2009) first defined the exponential type estimator (t_{exp1}) by adapting the estimator proposed by Bahl and Tuteja (1991) to the Case I. Following these estimators, Olufadi and Kumar (2014), Yadav et al. (2016), Kumar and Kumar (2017), Pal and Singh (2016, 2017, 2018), Dansawad (2019), Singh and Usman (2019a, 2019b), Unal and Kadilar (2021) proposed various estimators taking the advantage of the exponential function.

In these estimators, $\bar{y}^*$ is the sample mean of *y* under the non-response scheme. Also, $\bar{x}$ and $\bar{X}$ are the sample and the population means of *x*, respectively, while $\bar{Y}$ is the population mean of *y*.

Under the Case II, Cochran (1977) defined the ratio (t_{R2}) and regression (t_{reg2}) estimators while Singh et al. (2009) proposed the exponential type estimator for the population mean. Following these estimators, Kumar and Bhougal (2011), Kumar (2013), Yadav et al (2016), Kumar and Kumar (2017), Pal and Singh (2016, 2017, 2018), Singh and Usman (2019a, 2019b), Unal and Kadilar (2020, 2021) and Riaz et al. (2020) proposed estimators using the exponential function for the population mean for the Case II.

Note that in Tables 1 and 2, $\bar{x}^*$ refers the sample mean of *x* in the case of non-response and the coefficient of the population correlation between *x* and *y* for the non-response group is referred as $\rho_{yx(2)}$.

Table 1- Existing estimators in literature for the Case I

Authors	*Estimators*
Rao (1986)	$t_{R1} = \bar{y}^* \dfrac{\bar{X}}{\bar{x}}$
Rao (1986)	$t_{reg1} = \bar{y}^* + b^*\left(\bar{X} - \bar{x}\right),\ b^* = S_{xy}^* \big/ S_x^{*2}$
Singh et al. (2009)	$t_{\exp1} = \bar{y}^* exp\left(\dfrac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right)$
Olufadi and Kumar (2014)	$t_{YK1} = \bar{y}^* \left\{\alpha \exp\left(\dfrac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right) + (1-\alpha)\exp\left(\dfrac{\bar{x} - \bar{X}}{\bar{x} + \bar{X}}\right)\right\}$
Yadav et al. (2016)	$t_{Y1} = \bar{y}^* exp\left(\dfrac{\bar{X} - \bar{x}}{\bar{X} + (\alpha - 1)\bar{x}}\right)$
Pal and Singh (2016)	$t_{(\alpha,\delta),1} = \bar{y}^* \left(\dfrac{\bar{X}}{\bar{x}}\right)^{\alpha} exp\left\{\left(\dfrac{\delta\left(\bar{X} - \bar{x}\right)}{\bar{x} + \bar{X}}\right)\right\}$
Pal and Singh (2017)	$t_{PS1} = \alpha \bar{y}^* \left(\dfrac{\bar{X}}{\bar{x}}\right) + (1-\alpha)\bar{y}^* exp\left(\dfrac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right)$
Kumar and Kumar (2017)	$t_{KK1} = d_1^* \bar{y}^* exp\left(\dfrac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right)$
Pal and Singh (2018)	$t_{SP1} = \bar{y}^* \left[\eta\left\{\dfrac{(1-\delta)\bar{x} + \delta\bar{X}}{\delta\bar{x} + (1-\delta)\bar{X}}\right\} + (1-\eta)\left\{\dfrac{\delta\bar{x} + (1-\delta)\bar{X}}{(1-\delta)\bar{x} + \delta\bar{X}}\right\}\right]$
Dansawad (2019)	$t_D = \bar{y}^* exp\left(\dfrac{\left(a\bar{X} + b\right) - \left(a\bar{x} + b\right)}{\left(a\bar{X} + b\right) + \left(a\bar{x} + b\right)}\right)$
Singh and Usman (2019a)	$t_{US1} = \bar{y}^* \left(\dfrac{\bar{X}}{\bar{x}}\right)^{\alpha} exp\left\{\left(c\dfrac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right)\right\}$
Singh and Usman (2019b)	$t_{SU1} = \left[d_1^* \bar{y}^* + d_2^*\left(\bar{X} - \bar{x}\right)\right] exp\left(\dfrac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right)$
Unal and Kadilar (2021)	$t_{1,i} = k\bar{y}^* \left(\dfrac{a\bar{X} + b}{a\bar{x} + b}\right)^{c} \exp\left(\dfrac{a\left(\bar{X} - \bar{x}\right)}{a\left(\bar{X} + \bar{x}\right) + 2b}\right)$

Table 2- Existing estimators in literature for the Case II

Authors	*Estimators*
Cochran (1977)	$t_{R2}=\bar{y}^{*}\dfrac{\bar{X}}{\bar{x}^{*}}$
Cochran (1977)	$t_{reg2}=\bar{y}^{*}+b^{*}\left(\bar{X}-\bar{x}^{*}\right)$
Singh et al. (2009)	$t_{\exp 2}=\bar{y}^{*}exp\left(\dfrac{\bar{X}-\bar{x}^{*}}{\bar{X}+\bar{x}^{*}}\right)$
Kumar and Bhougal (2011)	$t_{KB}=\bar{y}^{*}\left\{\alpha\exp\left(\dfrac{\bar{X}-\bar{x}^{*}}{\bar{X}+\bar{x}^{*}}\right)+(1-\alpha)\exp\left(\dfrac{\bar{x}^{*}-\bar{X}}{\bar{x}^{*}+\bar{X}}\right)\right\}$
Kumar (2013)	$t_{K}=\bar{y}^{*}exp\left(\dfrac{\left(a\bar{X}+b\right)-\left(a\bar{x}^{*}+b\right)}{\left(a\bar{X}+b\right)+\left(a\bar{x}^{*}+b\right)}\right)$
Yadav et al. (2016)	$t_{Y2}=\bar{y}^{*}exp\left(\dfrac{\bar{X}-\bar{x}^{*}}{\bar{X}+(\alpha-1)\bar{x}^{*}}\right)$
Pal and Singh (2016)	$t_{(\alpha,\delta),2}=\bar{y}^{*}\left(\dfrac{\bar{X}}{\bar{x}^{*}}\right)^{\alpha}exp\left\{\left(\dfrac{\delta\left(\bar{X}-\bar{x}^{*}\right)}{\bar{X}+\bar{x}^{*}}\right)\right\}$
Pal and Singh (2017)	$t_{PS2}=\alpha\bar{y}^{*}\left(\dfrac{\bar{X}}{\bar{x}^{*}}\right)+(1-\alpha)\bar{y}^{*}exp\left(\dfrac{\bar{X}-\bar{x}^{*}}{\bar{X}+\bar{x}^{*}}\right)$
Kumar and Kumar (2017)	$t_{KK2}=d_{1}^{*}\bar{y}^{*}exp\left(\dfrac{\bar{X}-\bar{x}^{*}}{\bar{X}+\bar{x}^{*}}\right)$
Pal and Singh (2018)	$t_{SP2}=\bar{y}^{*}\left[\eta\left\{\dfrac{(1-\delta)\bar{x}^{*}+\delta\bar{X}}{\delta\bar{x}^{*}+(1-\delta)\bar{X}}\right\}+(1-\eta)\left\{\dfrac{\delta\bar{x}^{*}+(1-\delta)\bar{X}}{(1-\delta)\bar{x}^{*}+\delta\bar{X}}\right\}\right]$
Singh and Usman (2019a)	$t_{US2}=\bar{y}^{*}\left(\dfrac{\bar{X}}{\bar{x}^{*}}\right)^{\alpha}exp\left\{c\dfrac{\bar{X}-\bar{x}^{*}}{\bar{X}+\bar{x}^{*}}\right\}$
Singh and Usman (2019b)	$t_{SU2}=\left[d_{1}^{*}\bar{y}^{*}+d_{2}^{*}\left(\bar{X}-\bar{x}^{*}\right)\right]exp\left(\dfrac{\bar{X}-\bar{x}^{*}}{\bar{X}+\bar{x}^{*}}\right)$
Unal and Kadilar (2020)	$t_{UK}=\bar{y}^{*}\left(\dfrac{\bar{x}^{*}}{\bar{X}}\right)^{\alpha}\exp\left(\dfrac{\bar{X}-\bar{x}^{*}}{\bar{X}+\bar{x}^{*}}\right)$
Riaz et al. (2020)	$t_{RNAQ}=\bar{y}^{*}\left[d_{1}^{*}+d_{2}^{*}\left(a\bar{X}-a\bar{x}^{*}\right)\right]\exp\left(\dfrac{c\left(a\bar{X}-a\bar{x}^{*}\right)}{\left(a\bar{X}-a\bar{x}^{*}\right)+2b}\right)$
Unal and Kadilar (2021)	$t_{2,i}=k\bar{y}^{*}\left(\dfrac{a\bar{X}+b}{a\bar{x}^{*}+b}\right)^{c}\exp\left(\dfrac{a\left(\bar{X}-\bar{x}^{*}\right)}{a\left(\bar{X}+\bar{x}^{*}\right)+2b}\right)$

In Tables 1 – 2, α, d_1^*, d_2^*, k, η and δ are chosen constants that make MSE minimum and c takes the (0, -1, 1) values. Besides, a and b are either function of the known parameters of x or real numbers.

For this study, our main motivation is proposing more efficient estimator than existing estimators in literature in the presence of both non-response schemes as Case I and Case II. In this article, we introduce the proposed family of estimators for both cases. The appropriateness of the proposed family of estimators is examined theoretically and numerically on agriculture in Sections 4 and 5, respectively. Finally, the article is concluded with the obtained results.

2. Proposed Family of Estimators

Singh et al. (2020) proposed a new estimator using the exponential function. Based on this estimator, a new family of estimators for the population mean is proposed under the non-response scheme, Case I and Case II, respectively, as follows:

3.1. CASE I:

The first family of estimators is defined as

$$t_{C1,j} = \bar{y}^{*}\left(\nu_1 + \nu_2 \frac{\bar{X}}{\bar{x}}\right)\exp\left(\frac{\phi\left(\bar{X}-\bar{x}\right)}{\phi\left(\bar{X}+\bar{x}\right)+2\varphi}\right), j=1, 2, \ldots, 10, \tag{3}$$

Where; ν_1 and ν_2 are constants that make the $MSE\left(t_{C1,j}\right), j=1,2,\ldots,10$ minimum. Besides, ϕ and φ are either functions of the known parameters of x or real numbers.

In order to obtain the bias and MSE of the $t_{C1,j}, j=1,2,\ldots,10$, we use the following notations as

$$\bar{y}^{*} = \left(\bar{Y}e_y^{*} + \bar{Y}\right), \bar{x} = \left(\bar{X}e_x + \bar{X}\right), \qquad E\left(e_x\right)=0,\ E\left(e_x^2\right)=\lambda C_x^2,\ E\left(e_y^{*}\right)=0,\ E\left(e_y^{*2}\right)=\left(\lambda C_y^2 + \frac{W_2\left(z-1\right)}{n}C_{y(2)}^2\right)$$

$$E\left(e_y^{*}e_x\right)=\lambda C_{yx},$$

Where; $C_x^2 = \frac{S_x^2}{\bar{X}^2}, C_{xy} = \rho_{xy}C_xC_y,\ C_{x(2)}^2 = \frac{S_{x(2)}^2}{\bar{X}^2}, C_{yx(2)} = \rho_{yx(2)}C_{x(2)}C_{y(2)}$ and ρ_{xy} is the coefficient of the population correlation between y and x.

We re-write the Equation (3) using these notations as follows:

$$\begin{aligned} t_{C1,j} &= \bar{Y}\left(1+e_y^{*}\right)\left(\nu_1 + \nu_2 \frac{\bar{X}}{\bar{X}\left(1+e_x\right)}\right)\left(1-\vartheta e_x + \frac{3\vartheta^2}{2}e_x^2\right) \\ &= \bar{Y}\left(1+e_y^{*}\right)\left(\nu_1 - \nu_1\vartheta e_x + \frac{3\nu_1\vartheta^2}{2}e_x^2 + \nu_2 - \nu_2\vartheta e_x + \frac{3\nu_2\vartheta^2}{2}e_x^2 - \nu_2 e_x + \nu_2\vartheta e_x^2 + \nu_2 e_x^2\right), \end{aligned} \tag{4}$$

Where; $\vartheta = \frac{\phi\bar{X}}{2\left(\phi\bar{X}+\varphi\right)}$.

Using different ϕ and φ values, we can propose various estimators and some members of the proposed estimators are given in Table 3.

Table 3- Some members of the $t_{C1,j}$, $j=1,2,...,10$

Values			*Estimators*
ϑ_i	ϕ	φ	
$\vartheta_1=\dfrac{\bar{X}}{2(\bar{X}+1)}$	1	1	$t_{C1,1}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}}\right)\exp\left(\dfrac{(\bar{X}-\bar{x})}{(\bar{X}+\bar{x})+2}\right)$
$\vartheta_2=\dfrac{\bar{X}}{2(\bar{X}+\beta_2(x))}$	1	$\beta_2(x)$	$t_{C1,2}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}}\right)\exp\left(\dfrac{(\bar{X}-\bar{x})}{(\bar{X}+\bar{x})+2\beta_2(x)}\right)$
$\vartheta_3=\dfrac{\bar{X}}{2(\bar{X}+C_x)}$	1	C_x	$t_{C1,3}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}}\right)\exp\left(\dfrac{(\bar{X}-\bar{x})}{(\bar{X}+\bar{x})+2\varphi}\right)$
$\vartheta_4=\dfrac{\bar{X}}{2(\bar{X}+\rho)}$	1	ρ	$t_{C1,4}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}}\right)\exp\left(\dfrac{(\bar{X}-\bar{x})}{(\bar{X}+\bar{x})+2\rho}\right)$
$\vartheta_5=\dfrac{\beta_2(x)\bar{X}}{2(\beta_2(x)\bar{X}+C_x)}$	$\beta_2(x)$	C_x	$t_{C1,5}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}}\right)\exp\left(\dfrac{\beta_2(x)(\bar{X}-\bar{x})}{\beta_2(x)(\bar{X}+\bar{x})+2C_x}\right)$
$\vartheta_6=\dfrac{C_x\bar{X}}{2(C_x\bar{X}+\beta_2(x))}$	C_x	$\beta_2(x)$	$t_{C1,6}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}}\right)\exp\left(\dfrac{C_x(\bar{X}-\bar{x})}{C_x(\bar{X}+\bar{x})+2\beta_2(x)}\right)$
$\vartheta_7=\dfrac{C_x\bar{X}}{2(C_x\bar{X}+\rho)}$	C_x	ρ	$t_{C1,7}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}}\right)\exp\left(\dfrac{C_x(\bar{X}-\bar{x})}{C_x(\bar{X}+\bar{x})+2\rho}\right)$
$\vartheta_8=\dfrac{\rho\bar{X}}{2(\rho\bar{X}+C_x)}$	ρ	C_x	$t_{C1,8}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}}\right)\exp\left(\dfrac{\rho(\bar{X}-\bar{x})}{\rho(\bar{X}+\bar{x})+2C_x}\right)$
$\vartheta_9=\dfrac{\beta_2(x)\bar{X}}{2(\beta_2(x)\bar{X}+\rho)}$	$\beta_2(x)$	ρ	$t_{C1,9}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}}\right)\exp\left(\dfrac{\beta_2(x)(\bar{X}-\bar{x})}{\beta_2(x)(\bar{X}+\bar{x})+2\rho}\right)$
$\vartheta_{10}=\dfrac{\rho\bar{X}}{2(\rho\bar{X}+\beta_2(x))}$	ρ	$\beta_2(x)$	$t_{C1,10}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}}\right)\exp\left(\dfrac{\rho(\bar{X}-\bar{x})}{\rho(\bar{X}+\bar{x})+2\beta_2(x)}\right)$

We expand the right hand side of Equation (4) and after the terms, having powers of e_y^* and e_x two and higher, are neglected, we have

$$\left(t_{C1,j}-\bar{Y}\right)=\bar{Y}\left(\nu_1-\nu_1\vartheta e_x+\frac{3\nu_1\vartheta^2}{2}e_x^2+\nu_2-\nu_2\vartheta e_x+\frac{3\nu_2\vartheta^2}{2}e_x^2-\nu_2 e_x+\nu_2\vartheta e_x^2\right.$$
$$\left.+\nu_2 e_x^2+\nu_1 e_y^*-\nu_1\vartheta e_y^* e_x+\nu_2 e_y^*-\nu_2\vartheta e_y^* e_x-\nu_2 e_y^* e_x-1\right) \tag{5}$$

We take the expectation on both sides of Equation (5) to derive $B\left(t_{C1}\right)$ as

$$E\left(t_{C1,j}-\bar{Y}\right)=\bar{Y}\left(\left(\nu_1+\nu_2-1\right)+E\left(e_x^2\right)\left(\frac{3\vartheta^2}{2}\left(\nu_1+\nu_2\right)+\nu_2\left(\vartheta+1\right)\right)-E\left(e_y^* e_x\right)\left(\nu_1\vartheta+\nu_2\left(\vartheta+1\right)\right)\right)$$

$$B\left(t_{C1,j}\right)=\bar{Y}\left(\left(\nu_1+\nu_2-1\right)+\lambda C_x^2\left(\frac{3\vartheta^2}{2}\left(\nu_1+\nu_2\right)+\nu_2\left(\vartheta+1\right)\right)-\lambda C_{yx}\left(\nu_1\vartheta+\nu_2\left(\vartheta+1\right)\right)\right).$$

Expressions of the MSE are computed as follows:

$$\begin{aligned}\left(t_{C1,j}-\bar{Y}\right)^2=\bar{Y}^2\Big(1&+\nu_1^2\left(1+e_y^{*2}+4\vartheta^2e_x^2-4\vartheta e_y^*e_x\right)\\&+\nu_2^2\left(1+e_y^{*2}+3e_x^2+4\vartheta e_x^2+4\vartheta^2e_x^2-4\vartheta e_y^*e_x-4e_y^*e_x\right)\\&+\nu_1\nu_2\left(2+2e_y^{*2}+2e_x^2+4\vartheta e_x^2+8\vartheta^2e_x^2-8\vartheta e_y^*e_x-4e_y^*e_x\right)\\&-\nu_1\left(2+3\vartheta^2e_x^2-2\vartheta e_y^*e_x\right)\\&-\nu_2\left(2+3\vartheta^2e_x^2+2\vartheta e_x^2+2e_x^2-2\vartheta e_y^*e_x-2e_y^*e_x\right)\Big),\end{aligned}$$

$$\begin{aligned}E\left(t_{C1,j}-\bar{Y}\right)^2=\bar{Y}^2\Big(1&+\nu_1^2\left(1+E\left(e_y^{*2}\right)+4\vartheta^2E\left(e_x^2\right)-4\vartheta E\left(e_y^*e_x\right)\right)\\&+\nu_2^2\left(1+E\left(e_y^{*2}\right)+3E\left(e_x^2\right)+4\vartheta E\left(e_x^2\right)+4\vartheta^2E\left(e_x^2\right)-4\vartheta E\left(e_y^*e_x\right)-4E\left(e_y^*e_x\right)\right)\\&+\nu_1\nu_2\left(2+2E\left(e_y^{*2}\right)+2E\left(e_x^2\right)+4\vartheta E\left(e_x^2\right)+8\vartheta^2E\left(e_x^2\right)-8\vartheta E\left(e_y^*e_x\right)-4E\left(e_y^*e_x\right)\right)\\&-\nu_1\left(2+3\vartheta^2E\left(e_x^2\right)-2\vartheta E\left(e_y^*e_x\right)\right)\\&-\nu_2\left(2+3\vartheta^2E\left(e_x^2\right)+2\vartheta E\left(e_x^2\right)+2E\left(e_x^2\right)-2\vartheta E\left(e_y^*e_x\right)-2E\left(e_y^*e_x\right)\right)\Big),\end{aligned}$$

Where;

$$A_1=\left(1+E\left(e_y^{*2}\right)+4\vartheta^2E\left(e_x^2\right)-4\vartheta E\left(e_y^*e_x\right)\right)$$

$$A_2=\left(1+E\left(e_y^{*2}\right)+3E\left(e_x^2\right)+4\vartheta E\left(e_x^2\right)+4\vartheta^2E\left(e_x^2\right)-4\vartheta E\left(e_y^*e_x\right)-4E\left(e_y^*e_x\right)\right)$$

$$A_3=\left(1+\tfrac{3}{2}\vartheta^2E\left(e_x^2\right)-\vartheta E\left(e_y^*e_x\right)\right)$$

$$A_4=\left(1+\tfrac{3}{2}\vartheta^2E\left(e_x^2\right)+\vartheta E\left(e_x^2\right)+E\left(e_x^2\right)-\vartheta E\left(e_y^*e_x\right)-E\left(e_y^*e_x\right)\right)$$

$$A_5=\left(1+E\left(e_y^{*2}\right)+E\left(e_x^2\right)+2\vartheta E\left(e_x^2\right)+4\vartheta^2E\left(e_x^2\right)-4\vartheta E\left(e_y^*e_x\right)-2E\left(e_y^*e_x\right)\right)$$

and then we obtain $MSE\left(t_{C1,j}\right),\ j=1,2,...,10$ as

$$MSE\left(t_{C1,j}\right)=\bar{Y}^2\left(1+A_1\nu_1^2+A_2\nu_2^2+2A_5\nu_1\nu_2-2A_3\nu_1-2A_4\nu_2\right),\ j=1,2,...,10. \tag{6}$$

To obtain the minimum MSE of the $t_{C1,j},\ j=1,2,...,10$, we get the optimal values of ν_1 and ν_2, respectively, as follows:

$$\nu_1^*=\frac{A_2A_3-A_5A_4}{A_1A_2-A_5^2},\ \nu_2^*=\frac{A_1A_4-A_3A_5}{A_1A_2-A_5^2}.$$

Using the optimal values, ν_1^* and ν_2^*, and substituting them in Equation (6), we obtain

$$MSE_{\min}\left(t_{C1,j}\right)=\bar{Y}^2\left(1+A_1\nu_1^{*2}+A_2\nu_2^{*2}+2A_5\nu_1^*\nu_2^*-2A_3\nu_1^*-2A_4\nu_2^*\right) \tag{7}$$

$$=\bar{Y}^2\left(1-\frac{A_2A_3^2+A_1A_4^2-2A_3A_4A_5}{A_1A_2-A_5^2}\right),\ j=1,2,...,10. \tag{8}$$

3.2. CASE II:

The second family of estimators is proposed as

$$t_{C2,j}=\bar{y}^*\left(\nu_1+\nu_2\frac{\bar{X}}{\bar{x}^*}\right)\exp\left(\frac{\phi\left(\bar{X}-\bar{x}^*\right)}{\phi\left(\bar{X}+\bar{x}^*\right)+2\varphi}\right),\ j=1,2,\ldots,10. \tag{9}$$

Similarly, in the Case I, we re-write Equation (9) as

$$\begin{aligned}t_{C2,j}&=\bar{Y}\left(1+e_y^*\right)\left(\nu_1+\nu_2\frac{\bar{X}}{\bar{X}\left(1+e_x^*\right)}\right)\left(1-\vartheta e_x^*+\frac{3\vartheta^2}{2}e_x^{*2}\right),\\&=\bar{Y}\left(1+e_y^*\right)\left(\nu_1-\nu_1\vartheta e_x^*+\frac{3\nu_1\vartheta^2}{2}e_x^{*2}+\nu_2-\nu_2\vartheta e_x^*+\frac{3\nu_2\vartheta^2}{2}e_x^{*2}-\nu_2e_x^*+\nu_2\vartheta e_x^{*2}+\nu_2e_x^{*2}\right).\end{aligned} \tag{10}$$

We can write some members of the family of estimators for the Case II as in Table 4.

Expanding the right hand side of Equation (10) and then neglecting the terms having powers of e_y^* and e_x^* two and higher, we have

$$\begin{aligned}\left(t_{C2,j}-\bar{Y}\right)=\bar{Y}\Big(&\nu_1-\nu_1\vartheta e_x^*+\frac{3\nu_1\vartheta^2}{2}e_x^{*2}+\nu_2-\nu_2\vartheta e_x^*+\frac{3\nu_2\vartheta^2}{2}e_x^{*2}-\nu_2e_x^*+\nu_2\vartheta e_x^{*2}\\&+\nu_2e_x^{*2}+\nu_1e_y^*-\nu_1\vartheta e_y^*e_x^*+\nu_2e_y^*-\nu_2\vartheta e_y^*e_x^*-\nu_2e_y^*e_x^*-1\Big),\ j=1,2,\ldots,10.\end{aligned} \tag{11}$$

We take the expectation on both sides of Equation (11) as

$$E\left(t_{C2,j}-\bar{Y}\right)=\bar{Y}\left(\left(\nu_1+\nu_2-1\right)+E\left(e_x^{*2}\right)\left(\frac{3\vartheta^2}{2}\left(\nu_1+\nu_2\right)+\nu_2\left(\vartheta+1\right)\right)-E\left(e_y^*e_x^*\right)\left(\nu_1\vartheta+\nu_2\left(\vartheta+1\right)\right)\right).$$

Using $E\left(e_x^*\right)=0,\ E\left(e_y^*\right)=0$, $E\left(e_x^{*2}\right)=\lambda C_x^2+\frac{W_2\left(z-1\right)}{n}C_{x(2)}^2$, $E\left(e_y^*e_x^*\right)=\lambda\rho_{xy}C_xC_y+\frac{W_2\left(z-1\right)}{n}C_{xy(2)}$ $E\left(e_y^{*2}\right)=\lambda C_y^2+\frac{W_2\left(z-1\right)}{n}C_{y(2)}^2$ notations, we obtain the bias of the $t_{C2,j}$, $j=1,2,\ldots,10$ as follows:

Table 4- Some members of the $t_{C2,j}$, $j=1,2,\ldots,10$

Values			Estimators
ϑ_i	ϕ	φ	
$\vartheta_1=\dfrac{\bar{X}}{2(\bar{X}+1)}$	1	1	$t_{C2,1}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}^*}\right)\exp\left(\dfrac{(\bar{X}-\bar{x}^*)}{(\bar{X}+\bar{x}^*)+2}\right)$
$\vartheta_2=\dfrac{\bar{X}}{2(\bar{X}+\beta_2(x))}$	1	$\beta_2(x)$	$t_{C2,2}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}^*}\right)\exp\left(\dfrac{(\bar{X}-\bar{x}^*)}{(\bar{X}+\bar{x}^*)+2\beta_2(x)}\right)$
$\vartheta_3=\dfrac{\bar{X}}{2(\bar{X}+C_x)}$	1	C_x	$t_{C2,3}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}^*}\right)\exp\left(\dfrac{(\bar{X}-\bar{x}^*)}{(\bar{X}+\bar{x}^*)+2\varphi}\right)$
$\vartheta_4=\dfrac{\bar{X}}{2(\bar{X}+\rho)}$	1	ρ	$t_{C2,4}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}^*}\right)\exp\left(\dfrac{(\bar{X}-\bar{x}^*)}{(\bar{X}+\bar{x}^*)+2\rho}\right)$
$\vartheta_5=\dfrac{\beta_2(x)\bar{X}}{2(\beta_2(x)\bar{X}+C_x)}$	$\beta_2(x)$	C_x	$t_{C2,5}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}^*}\right)\exp\left(\dfrac{\beta_2(x)(\bar{X}-\bar{x}^*)}{\beta_2(x)(\bar{X}+\bar{x}^*)+2C_x}\right)$
$\vartheta_6=\dfrac{C_x\bar{X}}{2(C_x\bar{X}+\beta_2(x))}$	C_x	$\beta_2(x)$	$t_{C2,6}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}^*}\right)\exp\left(\dfrac{C_x(\bar{X}-\bar{x}^*)}{C_x(\bar{X}+\bar{x}^*)+2\beta_2(x)}\right)$
$\vartheta_7=\dfrac{C_x\bar{X}}{2(C_x\bar{X}+\rho)}$	C_x	ρ	$t_{C2,7}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}^*}\right)\exp\left(\dfrac{C_x(\bar{X}-\bar{x}^*)}{C_x(\bar{X}+\bar{x}^*)+2\rho}\right)$
$\vartheta_8=\dfrac{\rho\bar{X}}{2(\rho\bar{X}+C_x)}$	ρ	C_x	$t_{C2,8}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}^*}\right)\exp\left(\dfrac{\rho(\bar{X}-\bar{x}^*)}{\rho(\bar{X}+\bar{x}^*)+2C_x}\right)$
$\vartheta_9=\dfrac{\beta_2(x)\bar{X}}{2(\beta_2(x)\bar{X}+\rho)}$	$\beta_2(x)$	ρ	$t_{C2,9}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}^*}\right)\exp\left(\dfrac{\beta_2(x)(\bar{X}-\bar{x}^*)}{\beta_2(x)(\bar{X}+\bar{x}^*)+2\rho}\right)$
$\vartheta_{10}=\dfrac{\rho\bar{X}}{2(\rho\bar{X}+\beta_2(x))}$	ρ	$\beta_2(x)$	$t_{C2,10}=\bar{y}^*\left(\nu_1+\nu_2\dfrac{\bar{X}}{\bar{x}^*}\right)\exp\left(\dfrac{\rho(\bar{X}-\bar{x}^*)}{\rho(\bar{X}+\bar{x}^*)+2\beta_2(x)}\right)$

$$B(t_{C2,j})=\bar{Y}\Bigg((\nu_1+\nu_2-1)+\left(\lambda C_x^2+\frac{W_2(z-1)}{n}C_{x(2)}^2\right)\left(\frac{3\vartheta^2}{2}(\nu_1+\nu_2)+\nu_2(\vartheta+1)\right)$$
$$-\left(\lambda C_{yx}+\frac{W_2(z-1)}{n}C_{yx(2)}\right)\left(\nu_1\vartheta+\nu_2(\vartheta+1)\right)\Bigg),\ j=1,2,\ldots,10.$$

Similarly, expressions of the $MSE(t_{C2,j})$, $j=1,2,\ldots,10$ are computed, respectively, as follows:

$$\left(t_{C2,j}-\bar{Y}\right)^2=\bar{Y}^2\Big(1+\nu_1^2\left(1+e_y^{*2}+4\vartheta^2e_x^{*2}-4\vartheta e_y^*e_x^*\right)$$
$$+\nu_2^2\left(1+e_y^{*2}+3e_x^{*2}+4\vartheta e_x^{*2}+4\vartheta^2e_x^{*2}-4\vartheta e_y^*e_x^*-4e_y^*e_x^*\right)$$
$$+\nu_1\nu_2\left(2+2e_y^{*2}+2e_x^{*2}+4\vartheta e_x^{*2}+8\vartheta^2e_x^{*2}-8\vartheta e_y^*e_x^*-4e_y^*e_x^*\right)$$
$$-\nu_1\left(2+3\vartheta^2e_x^{*2}-2\vartheta e_y^*e_x^*\right)$$
$$-\nu_2\left(2+3\vartheta^2e_x^{*2}+2\vartheta e_x^{*2}+2e_x^{*2}-2\vartheta e_y^*e_x^*-2e_y^*e_x^*\right)\Big)$$

$$E\left(t_{C2,j}-\bar{Y}\right)^2=\bar{Y}^2\left(1+\nu_1^2\left(1+E\left(e_y^{*2}\right)+4\vartheta^2E\left(e_x^{*2}\right)-4\vartheta E\left(e_y^*e_x^*\right)\right)\right.$$
$$+\nu_2^2\left(1+E\left(e_y^{*2}\right)+3E\left(e_x^{*2}\right)+4\vartheta E\left(e_x^{*2}\right)+4\vartheta^2E\left(e_x^{*2}\right)-4\vartheta E\left(e_y^*e_x^*\right)-4E\left(e_y^*e_x^*\right)\right)$$
$$+\nu_1\nu_2\left(2+2E\left(e_y^{*2}\right)+2E\left(e_x^{*2}\right)+4\vartheta E\left(e_x^{*2}\right)+8\vartheta^2E\left(e_x^{*2}\right)-8\vartheta E\left(e_y^*e_x^*\right)-4E\left(e_y^*e_x^*\right)\right)$$
$$-\nu_1\left(2+3\vartheta^2E\left(e_x^{*2}\right)-2\vartheta E\left(e_y^*e_x^*\right)\right)$$
$$\left.-\nu_2\left(2+3\vartheta^2E\left(e_x^{*2}\right)+2\vartheta E\left(e_x^{*2}\right)+2E\left(e_x^{*2}\right)-2\vartheta E\left(e_y^*e_x^*\right)-2E\left(e_y^*e_x^*\right)\right)\right)$$

Where;

$$B_1=\left(1+E\left(e_y^{*2}\right)+4\vartheta^2E\left(e_x^{*2}\right)-4\vartheta E\left(e_y^*e_x^*\right)\right)$$
$$B_2=\left(1+E\left(e_y^{*2}\right)+3E\left(e_x^{*2}\right)+4\vartheta E\left(e_x^{*2}\right)+4\vartheta^2E\left(e_x^{*2}\right)-4\vartheta E\left(e_y^*e_x^*\right)-4E\left(e_y^*e_x^*\right)\right)$$
$$B_3=\left(1+{}^3\!/_2\,\vartheta^2E\left(e_x^{*2}\right)-\vartheta E\left(e_y^*e_x^*\right)\right)$$
$$B_4=\left(1+{}^3\!/_2\,\vartheta^2E\left(e_x^{*2}\right)+\vartheta E\left(e_x^{*2}\right)+E\left(e_x^{*2}\right)-\vartheta E\left(e_y^*e_x^*\right)-E\left(e_y^*e_x^*\right)\right)$$
$$B_5=\left(1+E\left(e_y^{*2}\right)+E\left(e_x^{*2}\right)+2\vartheta E\left(e_x^{*2}\right)+4\vartheta^2E\left(e_x^{*2}\right)-4\vartheta E\left(e_y^*e_x^*\right)-2E\left(e_y^*e_x^*\right)\right)$$

and then we obtain $MSE\left(t_{C2,j}\right),\ j=1,2,\ldots,10$ as

$$MSE\left(t_{C2,j}\right)=\bar{Y}^2\left(1+B_1\nu_1^2+B_2\nu_2^2+2B_5\nu_1\nu_2-2B_3\nu_1-2B_4\nu_2\right),\ j=1,2,\ldots,10. \tag{12}$$

The optimal values of ν_1 and ν_2 are obtained as

$$\nu_1^{**}=\frac{B_2B_3-B_5B_4}{B_1B_2-B_5^2},\ \nu_2^{**}=\frac{B_1B_4-B_3B_5}{B_1B_2-B_5^2}.$$

Using ν_1^{**} and ν_2^{**} and substituting them in Equation (22), we get

$$MSE_{\min}\left(t_{C2,j}\right)=\bar{Y}^2\left(1+B_1\nu_1^{**2}+B_2\nu_2^{**2}+2B_5\nu_1^{**}\nu_2^{**}-2B_3\nu_1^{**}-2B_4\nu_2^{**}\right)$$
$$=\bar{Y}^2\left(1-\frac{B_2B_3^2+B_1B_4^2-2B_3B_4B_5}{B_1B_2-B_5^2}\right),\ j=1,2,\ldots,10. \tag{13}$$

3. Efficiency Comparisons

One of the important features of an estimator is efficiency. After the theoretical inferences of the $t_{C1,j}$ and $t_{C2,j}$ *j=1, 2, …, 10* estimators, we obtain the efficiency comparisons for each case, separately. The conditions are given for the Case I and Case II in the next sub-sections.

a. Efficiency comparisons for the first case

We use the variance of the t_H and the MSE Equation of the t_{R1}, $t_{\exp 1}$, t_{reg1}, $t_{C1,j}$ *,j=1,2,…,10,* respectively, as follows:

$$MSE\left(t_{R1}\right)=\bar{Y}^2\left(\lambda\left(C_x^2-2C_{yx}+C_y^2\right)+\frac{W_2\left(z-1\right)}{n}C_{y(2)}^2\right), \tag{14}$$

$$MSE\left(t_{\exp1}\right)=\bar{Y}^2\left(\lambda\left(C_y^2+\frac{C_x^2}{4}-C_{yx}\right)+\frac{W_2\left(z-1\right)}{n}C_{y(2)}^2\right),$$

(15)

$$MSE\left(t_{reg1}\right)=\bar{Y}^2\left(\lambda C_y^2\left(1-\rho_{xy}^2\right)+\frac{W_2\left(z-1\right)}{n}C_{y(2)}^2\right) \tag{16}$$

and using Equations (2), (14), (15), and (16), we obtain that

- $\left[V\left(t_H\right)-MSE_{\min}\left(t_{C1,j}\right)\right]>0,\ j=1,2,...,10$

$$\left(\lambda C_y^2+\frac{W_2\left(z-1\right)}{n}C_{y(2)}^2\right)-\left(1-\frac{A_2A_3^2+A_1A_4^2-2A_3A_4A_5}{A_1A_2-A_5^2}\right)>0 \tag{17}$$

- $\left[MSE\left(t_{R1}\right)-MSE_{\min}\left(t_{C1,j}\right)\right]>0,\ j=1,2,...,10$

$$\left(\lambda\left(C_x^2-2C_{yx}+C_y^2\right)+\frac{W_2\left(z-1\right)}{n}C_{y(2)}^2\right)-\left(1-\frac{A_2A_3^2+A_1A_4^2-2A_3A_4A_5}{A_1A_2-A_5^2}\right)>0 \tag{18}$$

- $\left[MSE\left(t_{\exp1}\right)-MSE_{\min}\left(t_{C1,j}\right)\right]>0,\ j=1,2,...,10$

$$\left(\lambda\left(C_y^2+\frac{C_x^2}{4}-C_{yx}\right)+\frac{W_2\left(z-1\right)}{n}C_{y(2)}^2\right)-\left(1-\frac{A_2A_3^2+A_1A_4^2-2A_3A_4A_5}{A_1A_2-A_5^2}\right)>0 \tag{19}$$

- $\left[MSE\left(t_{reg1}\right)-MSE_{\min}\left(t_{C1,j}\right)\right]>0,\ j=1,2,...,10$

$$\left(\lambda C_y^2\left(1-\rho_{xy}^2\right)+\frac{W_2\left(z-1\right)}{n}C_{y(2)}^2\right)-\left(1-\frac{A_2A_3^2+A_1A_4^2-2A_3A_4A_5}{A_1A_2-A_5^2}\right)>0 \tag{20}$$

b. Efficiency comparisons for the second case

We use the variance of the t_H and the MSE Equation of the t_{R2}, $t_{\exp2}$, t_{reg2}, $t_{C2,j}$ *,j=1,2,...,10,* respectively, as follows:

$$MSE\left(t_{R2}\right)=\bar{Y}^2\left(\lambda\left(C_x^2-2C_{yx}+C_y^2\right)+\frac{W_2\left(z-1\right)}{n}\left(C_{y(2)}^2+C_{x(2)}^2-2C_{yx(2)}\right)\right), \tag{21}$$

$$MSE\left(t_{\exp2}\right)=\bar{Y}^2\left(\lambda C_y^2+\lambda\frac{C_x^2}{4}-\lambda C_{yx}+\frac{W_2\left(z-1\right)}{n}\left(C_{y(2)}^2+\frac{C_{x(2)}^2}{4}-C_{yx(2)}\right)\right), \tag{22}$$

$$MSE\left(t_{reg2}\right)=\bar{Y}^2\left(\lambda C_y^2\left(1-\rho_{xy}^2\right)+\frac{W_2\left(z-1\right)}{n}\left(C_{y(2)}^2+\rho_{xy}^2\frac{C_y^2}{C_x^2}C_{x(2)}^2-2\rho_{xy}\frac{C_y}{C_x}C_{yx(2)}\right)\right), \tag{23}$$

and using Equations (2), (21), (22) and (23), we obtain that

- $\left[V\left(t_H\right)-MSE_{\min}\left(t_{C2,j}\right)\right]>0,\ j=1,2,...,10$

$$\left(\lambda C_y^2+\frac{W_2\left(z-1\right)}{n}C_{y(2)}^2\right)-\left(1-\frac{B_2B_3^2+B_1B_4^2-2B_3B_4B_5}{B_1B_2-B_5^2}\right)>0 \tag{24}$$

- $\left[MSE\left(t_{R2}\right)-MSE_{\min}\left(t_{C2,j}\right)\right]>0,\ j=1,2,...,10$

$$\left(\lambda\left(C_x^2-2C_{yx}+C_y^2\right)+\frac{W_2\left(z-1\right)}{n}\left(C_{y(2)}^2+C_{x(2)}^2-2C_{yx(2)}\right)\right)-\left(1-\frac{B_2B_3^2+B_1B_4^2-2B_3B_4B_5}{B_1B_2-B_5^2}\right)>0 \tag{25}$$

- $\left[MSE\left(t_{\exp 2}\right)-MSE_{\min}\left(t_{C2,j}\right)\right]>0,\ j=1,2,...,10$

$$\left(\lambda C_y^2+\lambda\frac{C_x^2}{4}-\lambda C_{yx}+\frac{W_2(z-1)}{n}\left(C_{y(2)}^2+\frac{C_{x(2)}^2}{4}-C_{yx(2)}\right)\right)-\left(1-\frac{B_2B_3^2+B_1B_4^2-2B_3B_4B_5}{B_1B_2-B_5^2}\right)>0 \quad (26)$$

- $\left[MSE\left(t_{reg2}\right)-MSE_{\min}\left(t_{C2,j}\right)\right]>0,\ j=1,2,...,10$

$$\left(\lambda C_y^2\left(1-\rho_{xy}^2\right)+\frac{W_2(z-1)}{n}\left(C_{y(2)}^2+\rho_{xy}^2\frac{C_y^2}{C_x^2}C_{x(2)}^2-2\rho_{xy}\frac{C_y}{C_x}C_{yx(2)}\right)\right)-\left(1-\frac{B_2B_3^2+B_1B_4^2-2B_3B_4B_5}{B_1B_2-B_5^2}\right)>0 \quad (27)$$

The family of estimators, $t_{C1,j}$ and $t_{C2,j}$ j=1,2,…,10 are more efficient than others if the conditions for the efficiency, given in Equation (17) – Equation (20) and Equation (24) – Equation (27), are satisfied for both cases, respectively.

4. Empirical Studies

After theoretical inferences and comparisons, numerical illustrations are conducted in this section. The two real data sets are used to deal with the situation in which $t_{C1,j}$ and $t_{C2,j}$, $j=1,2,...,10$ estimators are more efficient than others. These data sets are also related with the agriculture for the purpose of showing the appropriateness of the proposed estimators, $t_{C1,j}$ and $t_{C2,j}$, $j=1,2,...,10$, in the agricultural field.

The parameter values for the Population 1 and Population 2 are given as follows:

Population 1. [Khare and Sinha (2009)]

In this population, the number of agriculture labors and the area of the village (in hectares) are considered as *y* and *x*, respectively. In this population, 25% of villages is greater than 160 ha and they symbolize the non-response group.

$N=96$	$n=40$	$\rho_{yx(2)}=0.72$	C_{yx}=0.8232	$C_{yx(2)}=1.4077$
$\bar{X}=144.87$	$W_2=0.25$	$\rho_{yx}=0.77$	$C_x=0.81$	$C_{x(2)}=0.94$
$\bar{Y}=137.92$	$\lambda=0.01458,\ f=0.42$	$\beta_2(x)=1.19$	$C_y=1.32$	$C_{y(2)}=2.08$

Population 2. [Khare and Srivastava (1993)]

In population 2, the cultivated area (in acres) and the population of the village are considered as *y* and *x*, respectively. In this population, 14 villages (i.e. 20% villages) are considered as non-response units.

$N=70$	$n=35$	$\rho_{yx(2)}=0.445$	$C_{yx}=0.3896$	$C_{yx(2)}=0.104$
$\bar{X}=1755.53$	$W_2=0.2$	$\rho_{yx}=0.778$	$C_x=0.801$	$C_{x(2)}=0.574$
$\bar{Y}=981.29$	$\lambda=0.0143,\ f=0.50$	$\beta_2(x)=0.34$	$C_y=0.6254$	$C_{y(2)}=0.4087$

In Tables 5 and 6, we present the MSE values of the $t_{C1,j}$, $j=1,2,...,10$ and some main estimators for the Populations 1 and 2, respectively. According to the results, we conclude that the $t_{C1,j}$, $j=1,2,...,10$ estimator has the minimum MSE value among other compared estimators using the different values of *z* for both populations.

Similarly, we present the MSE values of the $t_{C2,j}$, $j=1,2,...,10$ and other same estimators for the Populations 1 and 2, respectively, in Tables 7 and 8. According to the various values of *z*, we conclude that the $t_{C2,j}$, $j=1,2,...,10$ estimator has the minimum MSE value among others for both populations under the Case II as in the Case I.

Table 5- MSE values of the $t_{C1,j}$, $j=1,2,...,10$ and existing estimators for the Population I

Estimators	*z=2*	*z=3*	*z=4*	*z=5*	*z=6*
$t_{C1,1}$	686.3603	1152.707	1595.907	2017.642	2419.43350
$t_{C1,2}$	686.3668	1152.718	1595.922	2017.660	2419.45502
$t_{C1,3}$	686.3542	1152.697	1595.893	2017.624	2419.41298
$t_{C1,4}$	686.3529	1152.695	1595.890	2017.621	2419.40865
$t_{C1,5}$	686.3498	1152.690	1595.883	2017.612	2419.39836
$t_{C1,6}$	686.3758	1152.733	1595.942	2017.685	2419.48517
$t_{C1,7}$	686.3587	1152.704	1595.903	2017.637	2419.42817
$t_{C1,8}$	686.3620	1152.710	1595.911	2017.646	2419.43910
$t_{C1,9}$	686.3487	1152.688	1595.881	2017.609	2419.39474
$t_{C1,10}$	686.3782	1152.737	1595.948	2017.692	2419.49339
t_H	997.7000	1512.053	2026.406	2540.759	3055.11160
t_{R1}	722.9411	1237.294	1751.647	2265.999	2780.35271
t_{reg1}	711.1235	1225.476	1739.829	2254.182	2768.53507
t_{exp1}	814.8195	1329.172	1843.525	2357.878	2872.23111

Table 6- MSE values of the $t_{C1,j}$, $j=1,2,...,10$ and existing estimators for the Population II

Estimators	*z=2*	*z=3*	*z=4*	*z=5*	*z=6*
$t_{C1,1}$	3021.606751	3931.230876	4839.120003	5745.279089	6649.71308
$t_{C1,2}$	3021.605021	3931.228783	4839.117541	5745.276256	6649.70987
$t_{C1,3}$	3021.60623	3931.230246	4839.119261	5745.278236	6649.71211
$t_{C1,4}$	3021.60617	3931.230173	4839.119176	5745.278138	6649.71200
$t_{C1,5}$	3021.61030	3931.235167	4839.125048	5745.284896	6649.71965
$t_{C1,6}$	3021.60524	3931.229051	4839.117856	5745.276618	6649.71028
$t_{C1,7}$	3021.60668	3931.230786	4839.119896	5745.278967	6649.71294
$t_{C1,8}$	3021.60683	3931.230969	4839.120112	5745.279215	6649.71322
$t_{C1,9}$	3021.61012	3931.234955	4839.124798	5745.284609	6649.71933
$t_{C1,10}$	3021.60528	3931.229090	4839.117903	5745.276672	6649.71034
t_H	6299.48072	7218.587508	8137.694294	9056.801081	9975.90787
t_{R1}	4402.05564	5321.162427	6240.269214	7159.3760	8078.48279
t_{reg1}	3042.82647	3961.933251	4881.040038	5800.146825	6719.25361
t_{exp1}	3144.83018	4063.936966	4983.043753	5902.15054	6821.25733

Table 7- MSE values of the $t_{C2,j}$, $j=1,2,...,10$ and existing estimators for the Population I

Estimators	*z=2*	*z=3*	*z=4*	*z=5*	*z=6*
$t_{C2,1}$	444.1827	685.4467	921.5507	1153.376	1381.321333
$t_{C3,2}$	444.1895	685.4614	921.5759	1153.414	1381.37485
$t_{C2,3}$	444.1762	685.4328	921.5268	1153.339	1381.270296
$t_{C2,4}$	444.1748	685.4298	921.5217	1153.332	1381.259528
$t_{C3,5}$	444.1716	685.4228	921.5097	1153.314	1381.233925
$t_{C2,6}$	444.199	685.4819	921.6111	1153.467	1381.449836
$t_{C2,7}$	444.181	685.4431	921.5445	1153.366	1381.308086
$t_{C2,8}$	444.1844	685.4505	921.5573	1153.386	1381.335255
$t_{C2,9}$	444.1704	685.4204	921.5054	1153.307	1381.224929
$t_{C2,10}$	444.2016	685.4875	921.6207	1153.482	1381.470268
t_H	997.7000	1512.053	2026.4058	2540.7587	3055.111599
t_{R2}	493.2647	777.9412	1062.6176	1347.2940	1631.970482
t_{reg2}	456.5109	716.2512	975.9915	1235.732	1495.472056
t_{exp2}	673.7192	1046.972	1420.2242	1793.4767	2166.729263

Table 8- MSE values of the $t_{C2,j}$, $j=1,2,...,10$ and existing estimators for the Population II

Estimators	*z=2*	*z=3*	*z=4*	*z=5*	*z=6*
$t_{C2,1}$	2965.84881	3783.26051	4576.61636	5353.6645	6119.0051
$t_{C3,2}$	2965.84763	3783.25975	4576.61625	5353.6652	6119.0068
$t_{C2,3}$	2965.84846	3783.26028	4576.61633	5353.6647	6119.0055
$t_{C2,4}$	2965.84842	3783.26026	4576.61633	5353.6647	6119.0056
$t_{C3,5}$	2965.85124	3783.26205	4576.61658	5353.6630	6119.0014
$t_{C2,6}$	2965.84778	3783.25985	4576.61626	5353.6651	6119.0065
$t_{C2,7}$	2965.84876	3783.26048	4576.61636	5353.6645	6119.0051
$t_{C2,8}$	2965.84887	3783.26055	4576.61637	5353.6644	6119.0049
$t_{C2,9}$	2965.85112	3783.26198	4576.61657	5353.6630	6119.0016
$t_{C2,10}$	2965.84780	3783.25987	4576.61627	5353.6651	6119.0065
t_H	6299.48072	7218.58751	8137.69429	9056.8011	9975.9079
t_{R2}	5065.70210	6648.45535	8231.20860	9813.9618	11396.7150
t_{reg2}	3013.88002	3904.04037	4794.20072	5684.3611	6574.52141
t_{exp2}	3023.57939	3821.43539	4619.29140	5417.1474	6215.0034

The Percentage Relative Efficiency (PRE) values of $t_{C1,j}$, $j=1,2,...,10$ and $t_{C2,j}$, $j=1,2,...,10$ and existing estimators in literature for various values of *z* with respect to the Hansen and Hurwitz estimator (t_H) are presented in Tables 9 – 12 based on Populations 1 – 2, respectively, using the formulae as follows:

$$PRE(t) = \frac{MSE(t_H)}{MSE(t)} x100.$$

Table 9- PRE values of the $t_{C1,j}$, $j=1,2,...,10$ and existing estimators for the Population I

Estimators	*z=2*	*z=3*	*z=4*	*z=5*	*z=6*
$t_{C1,1}$	145.3609723	131.1740864	126.9751881	125.9271596	126.2738405
$t_{C1,2}$	145.3596104	131.1728691	126.9740256	125.9260232	126.2727173
$t_{C1,3}$	145.3622712	131.1752472	126.9762967	125.9282433	126.2749115
$t_{C1,4}$	145.3625452	131.1754922	126.9765306	125.928472	126.2751375
$t_{C1,5}$	145.3631968	131.1760745	126.9770867	125.9290156	126.2756748
$t_{C1,6}$	145.3577021	131.1711634	126.9723966	125.9244308	126.2711436
$t_{C1,7}$	145.3613094	131.1743877	126.9754758	125.9274409	126.2741185
$t_{C1,8}$	145.360618	131.1737697	126.9748857	125.926864	126.2735483
$t_{C1,9}$	145.3634257	131.1762791	126.9772821	125.9292066	126.2758636
$t_{C1,10}$	145.3571821	131.1706986	126.9719528	125.9239969	126.2707148
t_H	100	100	100	100	100
t_{R1}	138.0057057	122.2064342	115.6857459	112.1252828	109.8821594
t_{reg1}	140.2991209	123.3849076	116.4715312	112.713104	110.3511972
t_{exp1}	122.4442927	113.7589736	109.9201504	107.7561464	106.3671926

Table 10- PRE values of the $t_{C1,j}$, $j=1,2,...,10$ and existing estimators for the Population II

Estimators	*z=2*	*z=3*	*z=4*	*z=5*	*z=6*
$t_{C1,1}$	208.4811572	183.6215612	168.164755	157.6390101	150.020125
$t_{C1,2}$	208.4812766	183.6216589	168.1648405	157.6390878	150.0201974
$t_{C1,3}$	208.4811932	183.6215906	168.1647807	157.6390335	150.0201468
$t_{C1,4}$	208.4811973	183.621594	168.1647837	157.6390362	150.0201493
$t_{C1,5}$	208.4809126	183.6213607	168.1645796	157.6388507	150.0199766
$t_{C1,6}$	208.4812613	183.6216464	168.1648295	157.6390778	150.0201881
$t_{C1,7}$	208.4811624	183.6215654	168.1647587	157.6390134	150.0201281
$t_{C1,8}$	208.4811519	183.6215568	168.1647511	157.6390066	150.0201218
$t_{C1,9}$	208.4809247	183.6213707	168.1645883	157.6388586	150.0199839
$t_{C1,10}$	208.481259	183.6216446	168.1648279	157.6390764	150.0201867
t_H	100	100	100	100	100
t_{R1}	143.1031599	135.6580936	130.4061414	126.50266	123.487394
t_{reg1}	207.0272753	182.1986149	166.7204987	156.14779	148.4675002
t_{exp1}	200.31227	177.6254791	163.307703	153.4491711	146.2473469

According to the PRE values, the proposed family of estimators, $t_{C1,j}$, $j=1,2,...,10$, perform better than compared estimators, t_{R1}, t_{reg1} and $t_{\exp1}$, under the Case I for both populations.

Table 11- PRE values of the $t_{C2,j}$, $j=1,2,...,10$ and existing estimators for the Population I

Estimators	*z=2*	*z=3*	*z=4*	*z=5*	*z=6*
$t_{C2,1}$	224.6148091	220.5937908	219.8908604	220.2888855	221.1731279
$t_{C3,2}$	224.6113763	220.5890807	219.8848622	220.2816	221.1645593
$t_{C2,3}$	224.6180833	220.5982832	219.8965812	220.2958339	221.1813001
$t_{C2,4}$	224.6187742	220.5992311	219.8977883	220.2973	221.1830244
$t_{C3,5}$	224.6204171	220.6014849	219.9006584	220.300786	221.1871243
$t_{C2,6}$	224.6065675	220.582482	219.8764587	220.2713929	221.1525544
$t_{C2,7}$	224.6156589	220.5949568	219.8923452	220.290689	221.175249
$t_{C2,8}$	224.613916	220.5925654	219.8892999	220.2869901	221.1708987
$t_{C2,9}$	224.6209943	220.6022769	219.9016669	220.3020109	221.188565
$t_{C2,10}$	224.6052573	220.5806842	219.8741691	220.2686119	221.1492835
t_H	100	100	100	100	100
t_{R2}	202.2646312	194.3659785	190.6994416	188.5823463	187.2038515
t_{reg2}	218.5490019	211.1065133	207.6253587	205.6076222	204.2907847
t_{exp2}	148.0884119	144.421566	142.6821051	141.6666661	141.0010771

Table 12- PRE values of the $t_{C2,j}$, $j=1,2,...,10$ and existing estimators for the Population II

Estimators	*z=2*	*z=3*	*z=4*	*z=5*	*z=6*
$t_{C2,1}$	212.4006015	190.8033424	177.8102783	169.1701289	163.0315363
$t_{C3,2}$	212.4006865	190.8033807	177.8102828	169.170106	163.0314897
$t_{C2,3}$	212.4006271	190.8033539	177.8102797	169.170122	163.0315222
$t_{C2,4}$	212.40063	190.8033552	177.8102798	169.1701212	163.0315206
$t_{C3,5}$	212.4004277	190.8032646	177.8102699	169.1701767	163.0316328
$t_{C2,6}$	212.4006756	190.8033758	177.8102822	169.1701089	163.0314957
$t_{C2,7}$	212.4006052	190.8033441	177.8102785	169.1701279	163.0315342
$t_{C2,8}$	212.4005977	190.8033407	177.8102781	169.1701299	163.0315383
$t_{C2,9}$	212.4004363	190.8032685	177.8102703	169.1701743	163.031628
$t_{C2,10}$	212.400674	190.803375	177.8102822	169.1701094	163.0314965
t_H	100	100	100	100	100
t_{R2}	124.3555305	108.5754078	98.86390559	92.28486137	87.53318644
t_{reg2}	209.0156433	184.9004319	169.7403754	159.3283921	151.7358793
t_{exp2}	208.34514	188.8972797	176.1675893	167.1876435	160.5133131

Similarly in $t_{C1,j}$, $j=1,2,...,10$, it is shown that the proposed family of estimators, $t_{C2,j}$, $j=1,2,...,10$, perform better than compared estimators, t_{R2}, t_{reg2} and $t_{\exp2}$, under the Case II for both populations.

These results show that the proposed families of estimators $t_{C1,j}$ and $t_{C2,j}$, $j=1,2,...,10$ can be applied for estimating the population mean in the agriculture field for both cases.

5. Conclusions

This article proposes a family of estimators using the exponential function on estimation of the population mean in the presence of non-response. Firstly, we obtain the theoretical inferences and comparisons for the estimators under the Case I and Case II and then we found that the $t_{C1,j}$ and $t_{C2,j}$, $j=1,2,...,10$ estimators are more efficient than others in literature under the obtained conditions for both cases. In empirical studies, we use real data sets with the aim of showing the appropriateness of estimators in agriculture. According to the obtained results, the proposed family of estimators can appropriately be used in the agriculture on the estimation of the population mean.

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