# Dynamic Analysis Of Composite Sandwich Beams Under Moving Mass

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**ABSTRACT:** In this study, approximate analytical solution for the dynamic response of composite sandwich beams subjected to moving mass is presented. Using modal superposition, the equation of motion for the beam is derived in matrix form. Since coefficients of the matrix equation of motion are time-dependent, Newmark's method is employed for numerical solution. Effects of the lamina thickness and the fiber orientation on the beam deflection and the contact force between the beam and the mass are studied.

**Key Words:** *Composite sandwich beams; Moving mass; Modal superposition; Dynamic analysis*

# Hareketli Kütle Etkisi Altindaki Kompozit Sandviç Kirişlerin Dinamik Analizi

**ÖZET:** Bu çalışmada, hareketli kütle etkisi altındaki kompozit sandviç kirişlerin dinamik davranışı için yaklaşık bir analitik çözüm sunulmuştur. Modal süperpozisyon kullanılarak kiriş için hareket denklemi matris formda elde edilmiştir. Bu hareket denkleminin katsayıları zamana bağlı olduğundan, Newmark metodu kullanılarak sayısal çözüm yapılmıştır. Tabaka yüksekliği ve lif doğrultularının, kiriş çökmesi ve hareketli kütle ile kiriş arasında meydana gelen değme kuvveti üzerindeki etkileri incelenmiştir.

**Anahtar Kelimeler:** *Kompozit sandviç kirişler; Hareketli kütle; Modal süperpozisyon; Dinamik analiz*

## 1. INTRODUCTION

Dynamics of continuous elastic systems due to the passage of different types of moving loads is of a great importance in many diverse fields of engineering. Especially, in bridge engineering, dynamic effects of the moving vehicles on bridge structures has attracted much attention during the last three decades because of increasing use of heavy and high-speed vehicles as well as the development of high-performance materials which results in more slender bridge cross-sections.

It is well known inertial effects of a heavy vehicle travelling on an elastic structure are very important when it moves at high speeds [1-5]. In addition, separation between the mass and the supporting structure may occur in the case of greater vehicle to beam mass ratio [6-7]. Since equation of motion for the moving mass problem includes time-varying coefficients, a closed form solution is not available. Therefore, various approximate techniques have been used to solve the problem [1-9]. Bilello *et al.* [10] gave experimental validation of moving mass problem of a simply supported elastic beam.

Nowadays, the traditional heavy beams of simple materials are gradually being replaced by stronger composite beams with low weight. The use of composites in different engineering applications has tremendously increased because of their high strength, stiffness and favorable failure characteristics.

The vibration problem of composite structures due to different types of loads has been extensively studied [11-17]. Although dynamic analyses of isotropic structures under the action of moving loads were well studied, to the authors' knowledge, works on dynamic problem of laminated composite or sandwich beams under moving loads are rare [18-20].

This study is the extended version of the authors' previous paper [21]. It presents an approximate analytical solution of composite sandwich beams subjected to a moving mass by using equivalent mass and stiffness assumptions. The equation of motion with time-dependent coefficients for the beam is derived in matrix form by using the modal superposition. Deflection response of the beam and the interaction force between the beam and the mass are obtained numerically with using Newmark's method. Effects of the lamina thickness and the fiber orientation on the results are studied for two sandwich beam models.

## 2. FORMULATION

As shown in Figure 1, a multi-layered composite beam with simple supports is under the action of a lumped mass moving with a constant speed. In this model, the top and bottom layers (face sheets) are made from anisotropic composite material while the central (core) layer may be isotropic or anisotropic. The beam is initially at rest and it is assumed that mid-plane symmetry exists, i.e., the bending-stretching coupling and transverse shear are neglected.

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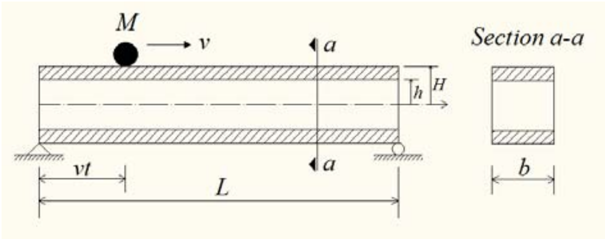


**Figure 1.** A simply supported sandwich beam under a mass moving with constant speed

Governing differential equation of motion of the problem can be written as follows:

$$(EI)_{eq}\frac{\partial^4 y(x,t)}{\partial x^4}+2m_{eq}\omega_b\frac{\partial y(x,t)}{\partial t}+ m_{eq}\frac{\partial^2 y(x,t)}{\partial t^2}=p(x,t) \quad (1)$$

where $(EI)_{eq}$, $m_{eq}$, $\omega_b$, $L$, and $y(x,t)$ represent the equivalent flexural rigidity, the equivalent mass per unit length, the circular frequency of damping, the length and the transverse deflection of the beam, respectively. Boundary and initial conditions are

$$y(0,t)=\frac{\partial^2 y(x,t)}{\partial x^2}\Big|_{x=0}= y(L,t)=\frac{\partial^2 y(x,t)}{\partial x^2}\Big|_{x=L}=0, \quad (2)$$

$$y(x,0)=\frac{\partial y(x,0)}{\partial t}\Big|_{t=0}=0 . \quad (3)$$

For the laminated composite beam shown in Figure 1, the equivalent mass per unit length and the stiffness can be written as [13, 19, 21].

$$m_{eq}=2b[\rho_c h+\rho_f(H-h)], \quad (4)$$

$$(EI)_{eq}=\frac{2b}{3}[E_c h^3+E_f(H^3-h^3)], \quad (5)$$

where $b$ is width of the beam, $H$ and $h$ are terms related with thickness as seen in Figure 1, $\rho$ is mass density and $E$ is Young's modulus. Subscripts "$c$" and "$f$" represent quantities belong to the core and face layers, respectively. $E_f$ can be written as

$$\frac{1}{E_f}=\frac{1}{E_{11}}\cos^4\theta+\left(\frac{1}{G_{12}}-\frac{2\nu_{12}}{E_{11}}\right)\cos^2\theta\sin^2\theta+\frac{1}{E_{22}}\sin^4\theta, \quad (6)$$

where $E_{11}$, $E_{22}$, $G_{12}$ and $\nu_{12}$ are the mechanical properties of face lamina along their principal directions. $\theta$ is the fiber angle measured from $x$- axis in counter-clock wise direction.

The load term $p(x,t)$ in Eq. (1) can be written as

$$p(x,t)=P(t)\delta(x-vt), \quad (7)$$

where $\delta(\cdots)$ is the Dirac delta function and $P(t)$ is defined as

$$P(t)=Mg-M\left(\frac{\partial^2 y(x,t)}{\partial t^2}+2v\frac{\partial^2 y(x,t)}{\partial x\,\partial t}+v^2\frac{\partial^2 y(x,t)}{\partial x^2}\right)\Bigg|_{x=vt}, \quad (8)$$

where $M$, $v$ and $g$ are the mass and speed of the moving load, and acceleration of gravity, respectively.

In modal form, transverse deflection $y(x,t)$ of the beam can be written as follows:

$$y(x,t)=\sum_{n=1}^{N}\phi_n(x)q_n(t), \quad (9)$$

where $N$ is number of modes to be included in numerical calculations and $q_n(t)$ are the generalized co-ordinates to be determined. $\phi_n(x)$ represents the normal modes of the beam. For simply supported beams, it can be defined as

$$\phi_n(x)=\sin\left(\frac{n\pi x}{L}\right). \quad (10)$$

Introducing the solution given by Eq. (9) into Eq. (1), multiplying the result by $\phi_i(x)$, and then integrating it over the interval $0\le x\le L$ yields

$$\sum_{n=1}^{N}\left\{\delta_{in}+2\frac{M}{mL}\phi_i(x=vt)\phi_n(x=vt)\right\}\ddot{q}_n(t)+ \sum_{n=1}^{N}\left\{2\xi_n\omega_n\delta_{in}+4\frac{M}{mL}v\phi_i(x=vt)\phi_n'(x=vt)\right\}\dot{q}_n(t)$$

$$+\sum_{n=1}^{N}\left\{\omega_n^2\delta_{in}+2\frac{M}{mL}v^2\phi_i(x=vt)\phi_n''(x=vt)\right\}q_n(t) =2\frac{Mg}{mL}\phi_i(x=vt) \quad i=1,2,3,\cdots,N \quad (11)$$

where $\delta_{in} = 1$ when $i = n$, and $\delta_{in} = 0$ when $i \neq n$. Dots and primes represent the derivatives with respect to time *t* and spatial coordinate *x*, respectively. $\xi_n$ and $\omega_n$ are the damping coefficient and the natural circular frequency for *n*th mode, respectively, and they can be defined as follows:

$$\xi_n = \frac{\omega_b}{\omega_n}, \quad (12)$$

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{(EI)_{eq}}{m_{eq}}}. \quad (13)$$

Eq. (11) indicates a set of coupled ordinary differential equations, and can be written in the following matrix form.

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{P}. \quad (14)$$

where, the matrices $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, and the vector $\mathbf{P}$ are time-varying and defined as

$$\mathbf{M} = \mathbf{I} + 2\frac{M}{mL} diag[\phi_i(x = vt)]\mathbf{\Phi},$$

$$\mathbf{C} = diag[2\xi_n\omega_n] + 4\frac{M}{mL} v\,diag[\phi_i(x = vt)]\mathbf{\Phi}',$$

$$\mathbf{K} = diag[\omega_n^2] + 2\frac{M}{mL} v^2 diag[\phi_i(x = vt)]\mathbf{\Phi}'',$$

$$\mathbf{P} = 2\frac{Mg}{mL}[\phi_1(x = vt), \phi_2(x = vt), \cdots, \phi_N(x = vt)]^T, \quad (15)$$

where,

$$\mathbf{\Phi} = \begin{bmatrix} \phi_1(x = vt) & \phi_2(x = vt) & \cdots & \phi_N(x = vt) \\ \phi_1(x = vt) & \phi_2(x = vt) & \cdots & \phi_N(x = vt) \\ \vdots & \vdots & & \vdots \\ \phi_1(x = vt) & \phi_2(x = vt) & \cdots & \phi_N(x = vt) \end{bmatrix}, \quad (16)$$

where, $\mathbf{I}$ is *n*-dimensional identity matrix, and the matrices $\mathbf{\Phi}'$ and $\mathbf{\Phi}''$ are the first and second derivatives of the matrix $\mathbf{\Phi}$ with respect to *x*. $diag[\cdots]$ indicates a diagonal matrix, e.g., $diag[\phi_i(x = vt)] = diag[\phi_1(x = vt), \phi_2(x = vt), \cdots, \phi_N(x = vt)]$.

The possibility of loss of contact between the mass and the beam can be examined by monitoring the contact force between the mass and the beam which is given by

$$F_c = Mg - M\left.\left(\frac{\partial^2 y(x,t)}{\partial t^2} + 2v\frac{\partial^2 y(x,t)}{\partial x \partial t} + v^2\frac{\partial^2 y(x,t)}{\partial x^2}\right)\right|_{x=vt}, \quad (17)$$

which is positive if acting in the downward direction. Changing of the interaction force from positive to negative indicates that the mass has separated from the beam and Eq. (1) is no longer valid to describe the ensuing motion.

## 3. RESULTS and DISCUSSION

Newmark's method is employed for numerical solution of Eq. (14). Table 1 shows material properties of two sandwich beam models used in the study. Model I has a softer core than that of Model II. Geometrical properties of the beam are $L = 4$ m, $b = 20$ cm and $2H = 40$ cm. In numerical calculations, damping is ignored and sufficient number of vibration modes is considered.

**Table 1**. Material properties for the beam models used in the study

| Model | Material | $\rho$ (kg/m$^3$) | $E_{11}$ (GPa) | $E_{22}$ (GPa) | $G_{12}$ (GPa) | $\upsilon_{12}$ |
|---|---|---|---|---|---|---|
| I | Carbon/Epoxy | 1600 | 177 | 10.8 | 76 | 0.270 |
| | Foam | 11.2 | 85.53 ×10$^{-5}$ | 85.53 ×10$^{-5}$ | 31.40 ×10$^{-5}$ | 0.362 |
| II | E-Glass/Epoxy | 2100 | 39 | 8.66 | 3.8 | 0.280 |
| | Balsa wood | 160 | 3.74 | 0.172 | 0.202 | 0.229 |

Figures 2 and 3 show variation of the fundamental frequency of the sandwich beam with the face lamina thickness to beam height ratio for different fiber angles. As $h_f / H$ goes to 1, the thickness of the face lamina increases. As seen in the figures, the fundamental frequency decreases with increasing the fiber angle. This is because an increase in the fiber angle causes a decrease in the beam stiffness. A linear relation is observed between $\omega_1 / 2\pi$ and $h_f / H$ for Model I (Figure 2) while not for Model II (Figure 3). When the stiffness of the core layer is so small compared to that of the face layers, it can, thus, be obtained a linear relation between the frequency and the thickness ratio.

In Figures 4 and 5, variation of maximum midpoint deflections of the beam with $h_f / H$ for different fiber angles is given. For Model I, the maximum midpoint deflections decrease with increasing $h_f / H$ (Figure 4). However, it is observed that the maximum midpoint deflections for Model II increase first and then decrease when $h_f / H$ increases (Figure 5). Curves in Figures 4 and 5 are compatible with those in Figures 2 and 3. As $h_f / H$ increases, stiffness of the beam also increases, and thus the beam deflections decrease. Figures 4 and 5 also show that the maximum beam deflections increase with increasing the fiber angle. Results in Figures 2 to 5 show that the natural frequency and the deflections of composite sandwich beams can be controlled by choosing the proper fiber angle or lamina thickness as previously reported in [19].

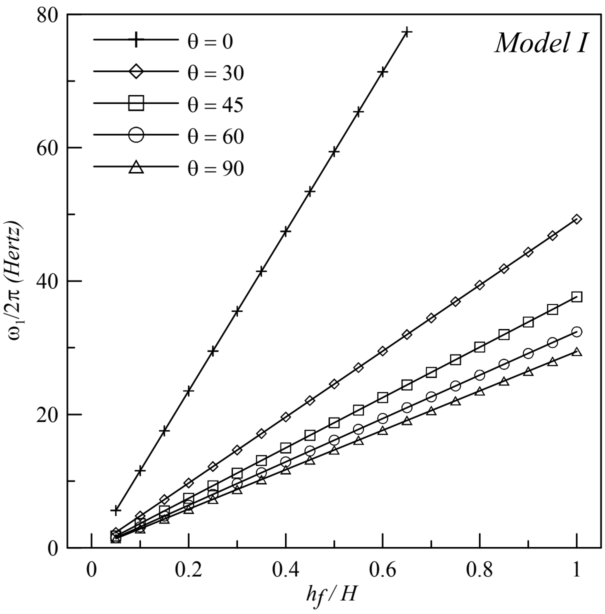


**Figure 2.** $\omega_1 / 2\pi$ *vs.* $h_f / H$ for different fiber angles

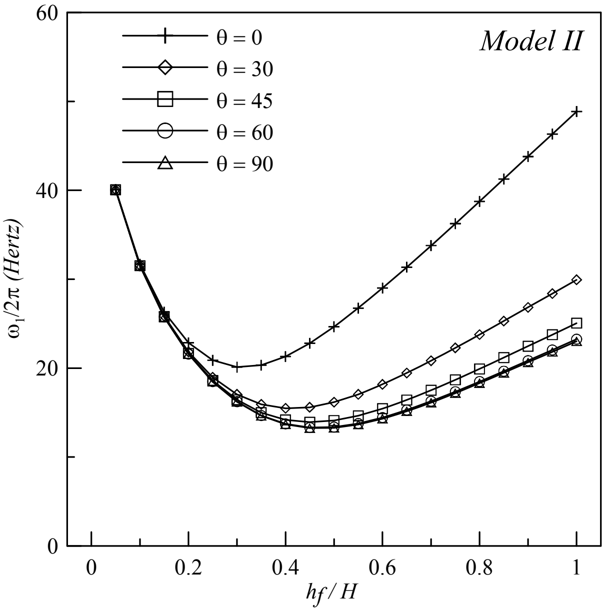


**Figure 3.** $\omega_1 / 2\pi$ *vs.* $h_f / H$ for different fiber angles

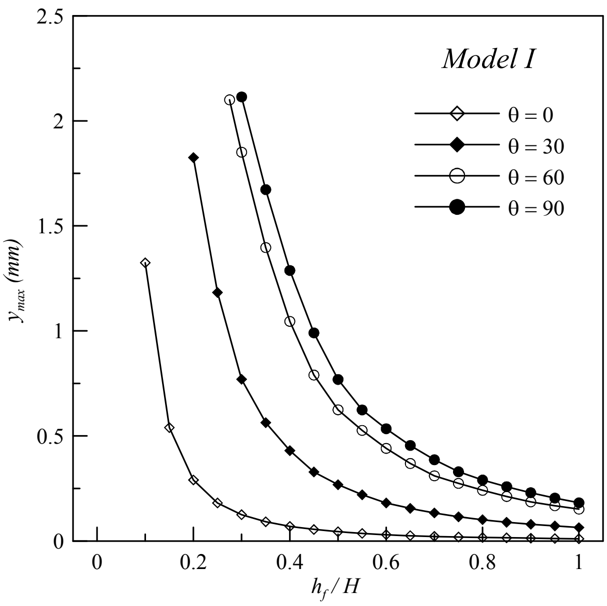


**Figure 4.** Maximum midpoint deflections *vs.* $h_f / H$ for different fiber angles, ( $M / mL = 0.3$ , $v = 50\, km / h$ )

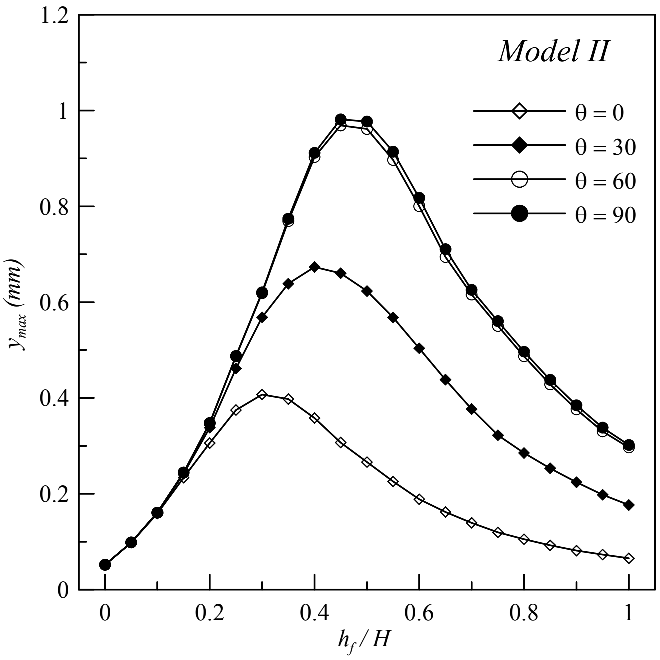


**Figure 5.** Maximum midpoint deflections *vs.* $h_f / H$ for different fiber angles, ( $M / mL = 0.3$ , $v = 50\ km / h$ )

Figures 6 and 7 give variation of beam deflections at midpoint for different fiber angles when the mass moves along the beam. From these figures, it is observed that midpoint deflections increase when the fiber angle increases due to decreasing of the beam stiffness. Deflection curves are observed to approach each other for $\theta > 45$ when core stiffness of the sandwich beam is increased.

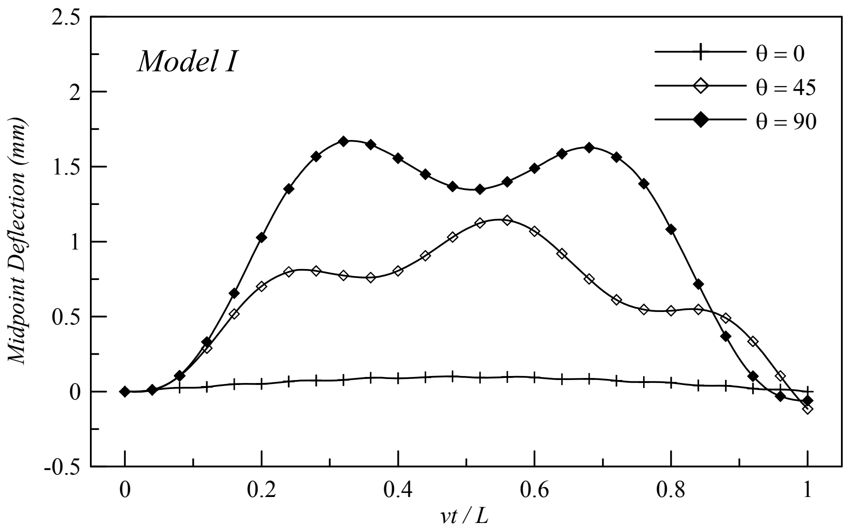


**Figure 6.** Midpoint deflections of the beam for different fiber angles ( $h_f / H = 1/3$ , $M / mL = 0.3$ , $v = 50\ km / h$ )

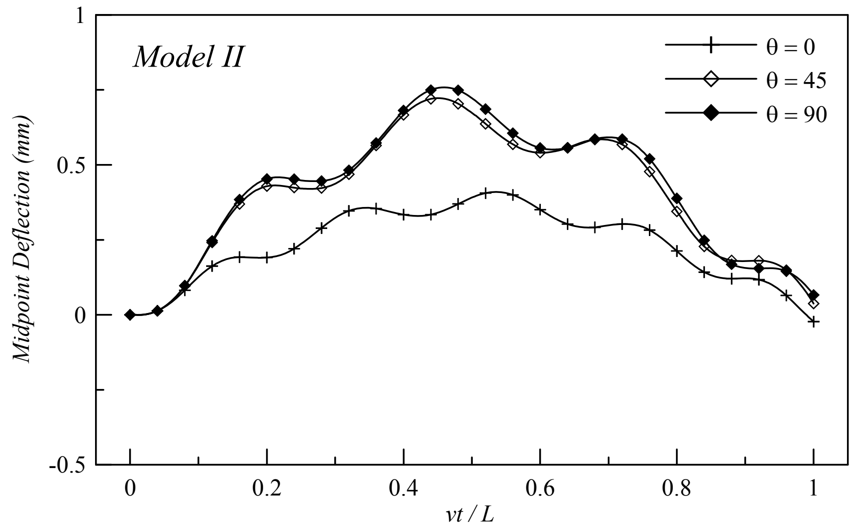


**Figure 7.** Midpoint deflections of the beam for different fiber angles ( $h_f / H = 1/3$, $M / mL = 0.3$, $v = 50\ km/h$ )

In Figures 8 and 9, the contact (interaction) force between the mass and the beam for different fiber angles is given. The contact force exhibits fluctuations when the mass travels along the beam. As the fiber angle increases, greater fluctuations are observed.

Figures 10 and 11 give a comparison of the midpoint deflections and the contact forces, respectively, for the beam models used. It is observed from these figures that smaller deflections but greater contact forces are obtained for Model II since the stiffness of the core layer is increased.

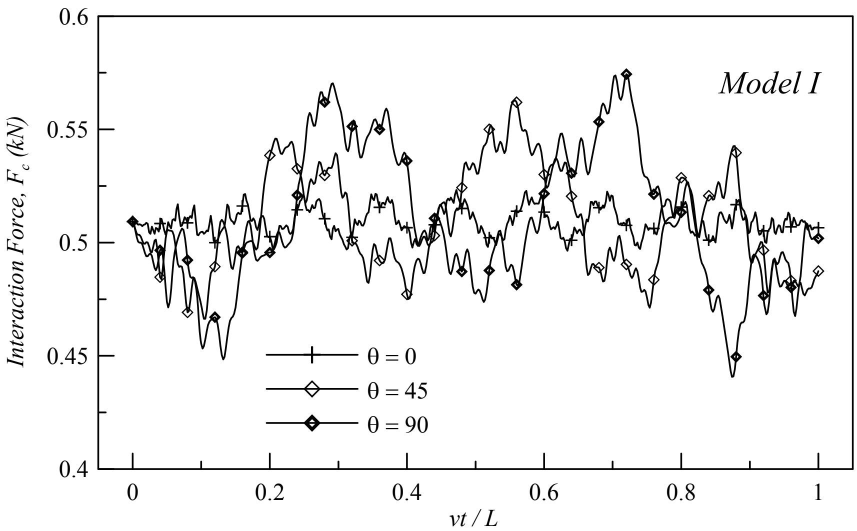


**Figure 8.** The contact force for different fiber angles ( $h_f / H = 1/3$, $M / mL = 0.3$, $v = 50\ km/h$ )

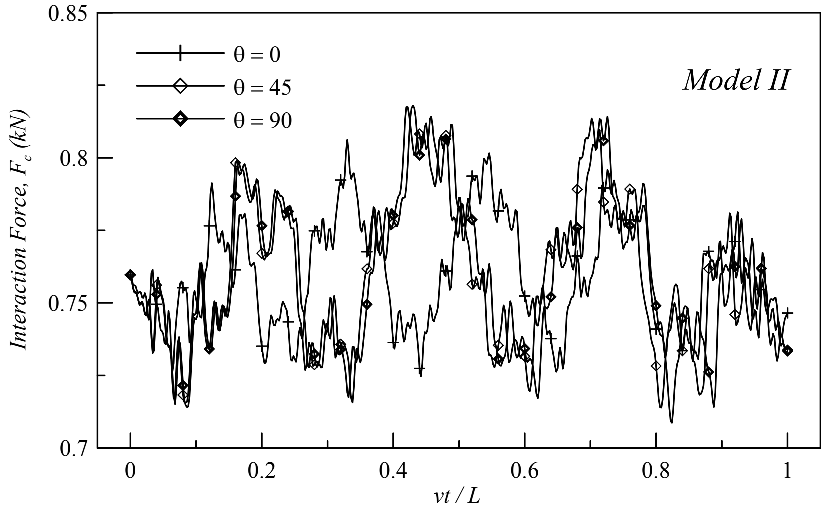


**Figure 9.** The contact force for different fiber angles ($h_f / H = 1/3$, $M / mL = 0.3$, $v = 50\ km/h$)

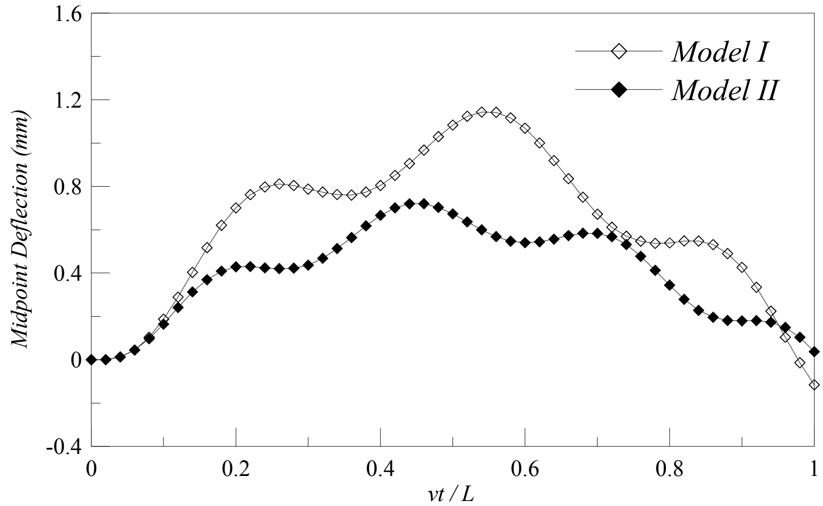


**Figure 10.** Comparison of midpoint deflections for the beam models used ($h_f / H = 1/3$, $\theta = 45$, $M / mL = 0.3$, $v = 50\ km/h$)

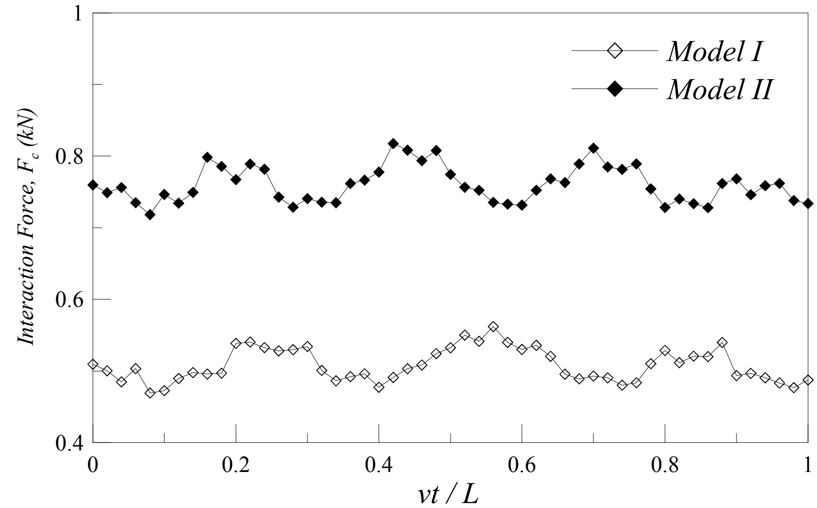


**Figure 11.** Comparison of the contact forces for the beam models used ($h_f / H = 1/3$, $\theta = 45$, $M / mL = 0.3$, $v = 50\ km/h$)

## 4. CONCLUSIONS

In this study, dynamic problem of composite sandwich beams subjected to a moving mass with constant speed is considered. The equation of motion with time-dependent coefficients for the beam is derived in matrix form by using the modal superposition and solved numerically by Newmark's direct integration scheme. Results show that increasing of the thickness and the fiber angle of the face lamina has a great effect on the deflection response of the beam and the contact force between the mass and the beam. For sandwich composite beams with soft-core layer, greater deflections but smaller contact forces are obtained. This type of beams is more flexible than those with hard-core layers. This study gives basic understanding about dynamics of the laminated composite beams. For better understanding, further investigations including the combined effect of anisotropy and transverse shear should be required.

## ACKNOWLEDGEMENTS

This study is supported by The Scientific and Technological Research Council of Turkey (TUBITAK) under International Postdoctoral Research Scholarship Program (2219).

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