Turk. J. Math. Comput. Sci.
15(2)(2023) 227–236
© MatDer
DOI : 10.47000/tjmcs.1195540

On a Topological Operator via Local Closure Function

Ayşe Nur Tunç[1,*], Sena Özen Yıldırım[1]

[1]*Department of Mathematics, Faculty of Sciences, Çanakkale Onsekiz Mart University, 17020, Çanakkale, Turkey.*

Received: 27-10-2022 • Accepted: 15-06-2023

Abstract. In this research, we define and study the new topological operator called Γ-boundary operator Bd^{Γ} by merging local closure function in ideal topological spaces. We research essential properties of this operator and we specialize Γ-boundary of some special sets, such as θ-open, $\mathfrak{I}_{\Gamma}$-perfect and $\mathfrak{I}_{\Gamma}$-dense. Moreover, we examine the properties of this operator in the topology which is formed by using local closure function. Furthermore, we compare Γ-boundary operator with the boundary operator and the $*$-boundary operator. We also show that under what conditions Γ-boundary operator, boundary operator and $*$-boundary operator are coincide.

2020 AMS Classification: 54A05, 54A99

Keywords: Operator Bd^{Γ}, local closure function, ideal topological space, $\mathfrak{I}_{\theta}$-open set.

1. Introduction

Operators like closure, interior and boundary operators [2] play a significant role in general topology. In 1966, Kuratowski defined the concept of the ideal [5] and introduced the concept of the local function [5] via ideal. Moreover, many topological operators was obtained by using local function, such as cl^* Kuratowski closure operator [15], Ψ-operator [8], the operator $()^{*-}$ [12], the operator $()^{*\Psi}$ [12] and $*$-boundary operator [12]. One of these operators which was studied by Selim et al. is $*$-boundary operator Bd^* [12]. Then, they characterized Hayashi-Samuel spaces and hence obtained new topology by using $*$-boundary operator in [12]. Furthermore, in [1] authors defined the concept of the local closure function and introduced the operator Ψ_{Γ} via local closure function. Then, they obtained the topologies σ_0 [1] and σ [1] by using the operator Ψ_{Γ}. In 2016, Pavlović obtained under what conditions local closure function and local function are coincide in [11]. In 2019, Goyal and Noorie defined the concepts of the θ-closure of a set with respect to an ideal [4] and $\mathfrak{I}_{\theta}$-closed set [4] via local closure function. Moreover, they produced a new topology $\tau_{\mathfrak{I}_{\theta}}$ [4] which is finer than τ_{θ} [16]. In addition to these studies, many authors considered the local closure function in detail (see [9, 10, 13, 14]). In this paper, we present new topological operator Bd^{Γ} by transforming the $*$-boundary operator via local closure function and we compare this operator with the boundary operator and the $*$-boundary operator. We also obtain some important properties of this operator and study the properties of Γ-boundary of some special sets.

*Corresponding Author
Email addresses: aysenurtunc@comu.edu.tr (A.N. Tunç), senaozen@comu.edu.tr (S. Özen Yıldırım)

2. Preliminaries

Throughout this article, (Z, τ) represents a topological space. In (Z, τ), the closure and the interior of a subset K of Z are denoted by $cl(K)$ and $int(K)$, respectively. $P(Z)$ represents the family of all subsets of Z. An ideal $\mathfrak{I}$ [5] on Z is a nonempty collection of subsets of Z satisfying the following conditions:

(i) if $K \in \mathfrak{I}$ and $L \subseteq K$, $L \in \mathfrak{I}$ (heredity),

(ii) if $K \in \mathfrak{I}$ and $L \in \mathfrak{I}$, $K \cup L \in \mathfrak{I}$ (finite additivity).

An ideal topological space $(Z, \tau, \mathfrak{I})$ is a topological space (Z, τ) with an ideal $\mathfrak{I}$ on Z. If $\tau \cap \mathfrak{I} = \{\emptyset\}$, then an ideal topological space $(Z, \tau, \mathfrak{I})$ is called Hayashi-Samuel space [3]. For a subset K of Z, $K^*(\mathfrak{I}, \tau) = \{x \in Z \mid U \cap K \notin \mathfrak{I}$ for each $U \in \tau(x)\}$ is called the local function [5] of K with respect to τ and $\mathfrak{I}$, where $\tau(x) = \{U \in \tau \mid x \in U\}$. We use K^* instead of $K^*(\mathfrak{I}, \tau)$. A Kuratowski closure operator $cl^*(.)$, for a topology $\tau^*(\mathfrak{I}, \tau)$, called the $*$-topology, is defined by $cl^*(K) = K \cup K^*(\mathfrak{I}, \tau)$ [15] and $\tau^*(\mathfrak{I}, \tau)$ is finer than τ. $\Gamma(K)(\mathfrak{I}, \tau) = \{x \in Z \mid K \cap cl(U) \notin \mathfrak{I}$ for every $U \in \tau(x)\}$ is called the local closure function [1] of K with respect to $\mathfrak{I}$ and τ. It is shortly denoted by $\Gamma(K)$ instead of $\Gamma(K)(\mathfrak{I}, \tau)$. An operator $\Psi_\Gamma : P(Z) \mapsto \tau$ is defined as $\Psi_\Gamma(K) = Z \setminus \Gamma(Z \setminus K)$ in [1]. A subset K is called $\mathfrak{I}_\Gamma$-perfect [13] (resp. Γ-dense-in-itself [13], L_Γ-perfect [13], R_Γ-perfect [13], $\mathfrak{I}_\Gamma$-dense [13]) if $K = \Gamma(K)$ (resp. $K \subseteq \Gamma(K)$, $K \setminus \Gamma(K) \in \mathfrak{I}$, $\Gamma(K) \setminus K \in \mathfrak{I}$, $\Gamma(K) = Z$). A subset K is called $\theta^{\mathfrak{I}}$-closed [10] if $\Gamma(K) \subseteq K$. Al-Omari and Noiri defined the topologies on Z in [1] as follows: $\sigma = \{K \subseteq Z : K \subseteq \Psi_\Gamma(K)\}$ and $\sigma_0 = \{K \subseteq Z : K \subseteq int(cl(\Psi_\Gamma(K)))\}$ and $\tau_\theta \subseteq \sigma \subseteq \sigma_0$. A subset K is called σ-open [1] (resp. σ_0-open [1]) set, if $K \in \sigma$ (resp. $K \in \sigma_0$).

For (Z, τ) and a subset K of Z, $cl_\theta(K) = \{x \in Z : cl(U) \cap K \neq \emptyset$ for each $U \in \tau(x)\}$ is called the θ-closure of K [16]. The θ-interior of K [16], denoted $int_\theta(K)$, consists of those points x of K such that $U \subseteq cl(U) \subseteq K$ for some open set U containing x. A subset K is called θ-closed [16] if $K = cl_\theta(K)$. The complement of a θ-closed set is called θ-open. The family of all θ-open sets in (Z, τ) is denoted by τ_θ. Moreover, τ_θ is a topology on Z and it is coarser than τ. A subset K is called preopen [7] if $K \subseteq int(cl(K))$. The complement of a preopen set is called a preclosed [7] set. A subset K is called generalized closed (briefly, g-closed) [6] if $cl(K) \subseteq U$, whenever $K \subseteq U$ and U is open.

In this paper, $(Z, \tau, \mathfrak{I})$ represents an ideal topological space.

Lemma 2.1 ([1]). (i) *In (Z, τ), $cl(O) = cl_\theta(O)$ for each open subset O of Z.*

(ii) *In $(Z, \tau, \mathfrak{I})$, $K^* \subseteq \Gamma(K)$ for $K \subseteq Z$.*

Theorem 2.2 ([1]). *The following features are valid for $M, N \subseteq Z$ in $(Z, \tau, \mathfrak{I})$.*

(i) $\Gamma(\emptyset) = \emptyset$.

(ii) *If $M \in \mathfrak{I}$, then* $\Gamma(M) = \emptyset$.

(iii) $\Gamma(M) \cup \Gamma(N) = \Gamma(M \cup N)$.

(iv) $\Psi_\Gamma(M \cap N) = \Psi_\Gamma(M) \cap \Psi_\Gamma(N)$.

(v) $\Gamma(M) = cl(\Gamma(M)) \subseteq cl_\theta(M)$.

Theorem 2.3 ([14]). *In $(Z, \tau, \mathfrak{I})$, $\Gamma(M \cap N) \subseteq \Gamma(M) \cap \Gamma(N)$ for $M, N \subseteq Z$.*

Definition 2.4 ([4]). In $(Z, \tau, \mathfrak{I})$ for a subset G of Z, θ-closure of G with respect to an ideal $\mathfrak{I}$ is defined as $cl_{\mathfrak{I}_\theta}(G) = G \cup \Gamma(G)(\mathfrak{I}, \tau)$ and if $G = cl_{\mathfrak{I}_\theta}(G)$, then G is called to be $\mathfrak{I}_\theta$-closed.

Remark 2.5 ([4]). In $(Z, \tau, \mathfrak{I})$ for a subset G of Z, $Int_{\mathfrak{I}_\theta}(G)$ is defined as $Int_{\mathfrak{I}_\theta}(G) = Z \setminus cl_{\mathfrak{I}_\theta}(Z \setminus G)$ and if $G = Int_{\mathfrak{I}_\theta}(G)$, then G is called to be $\mathfrak{I}_\theta$-open. The collection of $\mathfrak{I}_\theta$-open sets forms a topology on Z and it is denoted by $\tau_{\mathfrak{I}_\theta}$.

Remark 2.6. In $(Z, \tau, \mathfrak{I})$ for $M \subseteq Z$, M is $\mathfrak{I}_\theta$-closed $\Leftrightarrow M = cl_{\mathfrak{I}_\theta}(M) = \Gamma(M) \cup M \Leftrightarrow \Gamma(M) \subseteq M \Leftrightarrow M$ is $\theta^{\mathfrak{I}}$-closed. Thus, the concept of $\mathfrak{I}_\theta$-closed set in [4] and the concept of $\theta^{\mathfrak{I}}$-closed set in [10] are identical.

Proposition 2.7. *In $(Z, \tau, \mathfrak{I})$ for $M \subseteq Z$;*

(i) *M is $\mathfrak{I}_\theta$-open $\Leftrightarrow Z \setminus M$ is $\mathfrak{I}_\theta$-closed.*

(ii) *M is $\mathfrak{I}_\theta$-open $\Leftrightarrow M \subseteq \Psi_\Gamma(M)$.*

(iii) *M is σ-open $\Leftrightarrow M$ is $\mathfrak{I}_\theta$-open.*

Proof. *(i)* M is $\mathfrak{I}_\theta$-open $\Leftrightarrow M = Int_{\mathfrak{I}_\theta}(M) = Z \setminus cl_{\mathfrak{I}_\theta}(Z \setminus M) \Leftrightarrow cl_{\mathfrak{I}_\theta}(Z \setminus M) = Z \setminus M \Leftrightarrow Z \setminus M$ is $\mathfrak{I}_\theta$-closed.

(ii) M is $\mathfrak{I}_\theta$-open $\Leftrightarrow Z \setminus M$ is $\mathfrak{I}_\theta$-closed (or $\theta^{\mathfrak{I}}$-closed) $\Leftrightarrow \Gamma(Z \setminus M) \subseteq Z \setminus M \Leftrightarrow M \subseteq Z \setminus \Gamma(Z \setminus M) = \Psi_\Gamma(M)$.

(iii) The proof is clear. □

Corollary 2.8. *In $(Z, \tau, \mathfrak{I})$, $\sigma = \tau_{\mathfrak{I}_\theta}$ from the Proposition 2.7* (iii).

Remark 2.9. In $(Z, \tau, \mathfrak{I})$ for $K \subseteq Z$, $cl_{\mathfrak{I}_\theta}(K)$ may not be $\mathfrak{I}_\theta$-closed. Therefore, $cl_{\mathfrak{I}_\theta}$ is not a Kuratowski closure operator.

Example 2.10. Let $Z = \{p, q, r, s\}$, $\mathfrak{I} = \{\emptyset, \{p\}\}$ and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Z\}$. In $(Z, \tau, \mathfrak{I})$, $cl_{\mathfrak{I}_\theta}(cl_{\mathfrak{I}_\theta}(C)) \neq cl_{\mathfrak{I}_\theta}(C)$, for the set $C = \{r\}$.

Theorem 2.11 ([1]). *In $(Z, \tau, \mathfrak{I})$, $Z = \Gamma(Z)$ iff $cl(\tau) \cap \mathfrak{I} = \{\emptyset\}$ where $cl(\tau) = \{cl(G) : G \in \tau\}$.*

Theorem 2.12 ([14]). *$\Psi_\Gamma(K) \subseteq \Gamma(K)$ for each $K \subseteq Z$ in $(Z, \tau, \mathfrak{I})$ where $cl(\tau) \cap \mathfrak{I} = \{\emptyset\}$.*

Theorem 2.13. *In $(Z, \tau, \mathfrak{I})$, there is a subset M of Z such that $\Psi_\Gamma(M) = \Gamma(M)$ iff $cl(\tau) \cap \mathfrak{I} = \{\emptyset\}$.*

Proof. $(\Rightarrow)$: Let $M \subseteq Z$ such that $\Psi_\Gamma(M) = \Gamma(M)$. Then $Z \setminus \Gamma(Z \setminus M) = \Gamma(M)$ and so $Z = \Gamma(Z \setminus M) \cup \Gamma(M)$. Thus, by the Theorem 2.2 *(iii)*, $\Gamma(Z) = Z$. Consequently, from the Theorem 2.11, $cl(\tau) \cap \mathfrak{I} = \{\emptyset\}$.

$(\Leftarrow)$: Let $cl(\tau) \cap \mathfrak{I} = \{\emptyset\}$. From the Theorem 2.11, $\Gamma(Z) = Z$ and so $\Psi_\Gamma(Z) = Z \setminus \Gamma(\emptyset)$. In that case, by the Theorem 2.2 *(i)*, $\Psi_\Gamma(Z) = Z$ and thus $\Psi_\Gamma(Z) = \Gamma(Z)$. □

Theorem 2.14. *In $(Z, \tau, \mathfrak{I})$, if there is a subset M of Z with $\Psi_\Gamma(M) \neq \Gamma(M)$, then one of the following statements hold:*

***(a)** There exist $x \in Z$ and $U \in \tau(x)$ such that $U \in \mathfrak{I} \cap \tau(x)$.*

***(b)** There exists $x \in Z$ such that $cl(U) \notin \mathfrak{I}$ for every $U \in \tau(x)$.*

Proof. Let M be a subset of Z with $\Psi_\Gamma(M) \neq \Gamma(M)$. Afterward, there exists an element x of Z in either $\Psi_\Gamma(M) \setminus \Gamma(M)$ or $\Gamma(M) \setminus \Psi_\Gamma(M)$.

(a) If $x \in \Psi_\Gamma(M) \setminus \Gamma(M)$, $x \notin \Gamma(Z \setminus M)$ and $x \notin \Gamma(M)$. Therefore, there exist $G, H \in \tau(x)$ with $cl(G) \cap (Z \setminus M) \in \mathfrak{I}$ and $cl(H) \cap M \in \mathfrak{I}$. Let $U = G \cap H$. Hence, there exists $U \in \tau(x)$ such that $cl(U) \cap (Z \setminus M) \in \mathfrak{I}$ and $cl(U) \cap M \in \mathfrak{I}$. Then, $[cl(U) \cap (Z \setminus M)] \cup [cl(U) \cap M] = cl(U) \in \mathfrak{I}$. Consequently, $U \in \mathfrak{I}$ by the heredity.

(b) If $x \in \Gamma(M) \setminus \Psi_\Gamma(M)$, $x \in \Gamma(Z \setminus M)$ and $x \in \Gamma(M)$. By the Theorem 2.2 *(iii)*, $x \in \Gamma(Z \setminus M) \cup \Gamma(M) = \Gamma(Z)$. It implies that $cl(U) \cap Z = cl(U) \notin \mathfrak{I}$ for every $U \in \tau(x)$. □

3. The New Operator Bd^Γ

Definition 3.1. The operator $Bd^\Gamma : P(Z) \to \tau^k$, $Bd^\Gamma(K) = \Gamma(K) \cap \Gamma(Z \setminus K)$ is called Γ-boundary operator on $(Z, \tau, \mathfrak{I})$, where $\tau^k = \{K \subseteq Z : Z \setminus K \in \tau\}$. For $K \subseteq Z$ and $x \in Z$, a point $x \in Bd^\Gamma(K)$ is called a Γ-boundary point of K and $Bd^\Gamma(K)$ is called a Γ-boundary of K in $(Z, \tau, \mathfrak{I})$.

Example 3.2. Let $\mathbb{R}$ be the set of all real numbers, $\mathbb{Q}$ be the set of all rational numbers and τ_u be the usual topology on $\mathbb{R}$. In the ideal topological space $(\mathbb{R}, \tau_u, \{\emptyset\})$, $\Gamma(\mathbb{Q}) = \mathbb{R}$ and $\Gamma(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$ and so $Bd^\Gamma(\mathbb{Q}) = \mathbb{R}$.

Remark 3.3. In $(Z, \tau, \mathfrak{I})$, for a subset K of Z, Γ-boundary of K depends on both topology τ and ideal $\mathfrak{I}$. For example, in an ideal topological space $(\mathbb{R}, \tau, \{\emptyset\})$, where τ is discrete topology, $Bd^\Gamma(\mathbb{Q}) = \emptyset$. But we know $Bd^\Gamma(\mathbb{Q}) = \mathbb{R}$ in $(\mathbb{R}, \tau_u, \{\emptyset\})$ by the above example.

Example 3.4. Let $Z = \{p, q, r, s\}$, $\mathfrak{I}_1 = \{\emptyset, \{r\}\}$, $\mathfrak{I}_2 = \{\emptyset, \{p\}\}$ and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Z\}$. In $(Z, \tau, \mathfrak{I}_1)$, if $G = \{p, q, s\}$, then $Bd^\Gamma(G) = \emptyset$, but $Bd^\Gamma(G) = \{p, q, r\}$ in $(Z, \tau, \mathfrak{I}_2)$.

Proposition 3.5. *In $(Z, \tau, \mathfrak{I})$ for $K \subseteq Z$;*

(i) *If $\mathfrak{I} = \{\emptyset\}$, then $Bd^\Gamma(K) = cl_\theta(K) \cap cl_\theta(Z \setminus K)$.*

(ii) *If $\mathfrak{I} = P(Z)$, then $Bd^\Gamma(K) = \emptyset$.*

Proof. The proof is clear. □

Theorem 3.6. *In $(Z, \tau, \mathfrak{I})$, $Bd^\Gamma(K) = \Gamma(K) \setminus \Psi_\Gamma(K)$ for $K \subseteq Z$.*

Proof. $Bd^\Gamma(K) = \Gamma(K) \cap [Z \setminus (Z \setminus \Gamma(Z \setminus K))] = \Gamma(K) \cap (Z \setminus \Psi_\Gamma(K)) = \Gamma(K) \setminus \Psi_\Gamma(K)$. □

Theorem 3.7. *In $(Z, \tau, \mathfrak{I})$ for $K \subseteq Z$, if x is a Γ-boundary point of K, then $cl(U) \notin \mathfrak{I}$ for all $U \in \tau(x)$. But the reverse of this requirement is not true in general.*

Proof. Let $x \in Bd^\Gamma(K)$. Then, $x \in \Gamma(Z \setminus K)$ and $x \in \Gamma(K)$. By the Theorem 2.2 *(iii)*, $x \in \Gamma(Z \setminus K) \cup \Gamma(K) = \Gamma(Z)$. It implies that $cl(U) \cap Z = cl(U) \notin \mathfrak{I}$ for every $U \in \tau(x)$. □

Example 3.8. Let $Z = \{p, q, r, s\}$, $\mathfrak{I} = \{\emptyset, \{p\}\}$ and $\tau = \{\emptyset, \{p\}, \{s\}, \{p, q\}, \{p, s\}, \{p, q, s\}, Z\}$. In $(Z, \tau, \mathfrak{I})$, if $K = \{q\}$, then $Bd^\Gamma(K) = \{p, q, r\}$. Although $cl(U) \notin \mathfrak{I}$ for all $U \in \tau(s)$, $s \notin Bd^\Gamma(K)$.

Theorem 3.9. *In $(Z, \tau, \Im)$ for $K \subseteq Z$ and $x \in Z$, x is a Γ-boundary point of K iff for every $U \in \tau(x)$, $cl(U) \cap K \notin \Im$ and $cl(U) \cap (Z \setminus K) \notin \Im$.*

Proof. $x \in Bd^{\Gamma}(K) \Leftrightarrow x \in \Gamma(K)$ and $x \in \Gamma(Z \setminus K) \Leftrightarrow cl(U) \cap K \notin \Im$ and $cl(U) \cap (Z \setminus K) \notin \Im$ for every $U \in \tau(x)$. □

Theorem 3.10. *In $(Z, \tau, \Im)$ for $K \subseteq Z$, $Bd^{\Gamma}(K) = \varnothing$ iff $\Gamma(K) \subseteq \Psi_{\Gamma}(K)$.*

Proof. $Bd^{\Gamma}(K) = \varnothing \Leftrightarrow \Gamma(K) \subseteq Z \setminus \Gamma(Z \setminus K) = \Psi_{\Gamma}(K)$. □

Theorem 3.11. *Let $cl(\tau) \cap \Im = \{\varnothing\}$ in $(Z, \tau, \Im)$. Then $Bd^{\Gamma}(K) = \varnothing$ iff $\Gamma(K) = \Psi_{\Gamma}(K)$ for $K \subseteq Z$.*

Proof. Let $cl(\tau) \cap \Im = \{\varnothing\}$. Then we know that by the Theorem 2.12, $\Psi_{\Gamma}(K) \subseteq \Gamma(K)$ for each $K \subseteq Z$. Therefore, the proof is obvious from the Theorem 3.10. □

Corollary 3.12. *In $(Z, \tau, \Im)$ for $K \subseteq Z$, if $Bd^{\Gamma}(K) = K$, then $cl(U) \notin \Im$ for each $x \in K$ and for each $U \in \tau(x)$.*

Proof. It is clear by the Theorem 3.7. □

Remark 3.13. The reverse of the Corollary 3.12 may not be true in general.

Example 3.14. For $(Z, \tau, \Im)$ in the Example 2.10, if $D = \{s\}$, then $cl(U) \notin \Im$ for each $U \in \tau(s)$, but $Bd^{\Gamma}(D) = \{q, s\} \neq D$.

Corollary 3.15. *In $(Z, \tau, \Im)$, if there is a nonempty subset K of Z such that $Bd^{\Gamma}(K) = K$, $Z \notin \Im$, that is, $\Im \neq P(Z)$.*

Proof. It is trivial by the Corollary 3.12. □

Theorem 3.16. *If $Bd^{\Gamma}(K) = Z$, then both K and $Z \setminus K$ are $\Im_{\Gamma}$-dense for $K \subseteq Z$ in $(Z, \tau, \Im)$.*

Proof. Let $K \subseteq Z$ such that $Bd^{\Gamma}(K) = Z$. It implies that $\Gamma(K) = Z$ and $\Gamma(Z \setminus K) = Z$. As a consequence, both K and $Z \setminus K$ are $\Im_{\Gamma}$-dense. □

Theorem 3.17. *In $(Z, \tau, \Im)$, the followings hold for $K, L \subseteq Z$:*

(a) *$Bd^{\Gamma}(\varnothing) = \varnothing$.*
(b) *$Bd^{\Gamma}(Z) = \varnothing$.*
(c) *If $K \in \Im$, then $Bd^{\Gamma}(K) = \varnothing$.*
(d) *$Bd^{\Gamma}(K \cup L) \subseteq Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L)$.*
(e) *$(K \cap Bd^{\Gamma}(L)) \cup Bd^{\Gamma}(K \cup L) \cup (L \cap Bd^{\Gamma}(K)) \subseteq Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L)$.*
(f) *If $Bd^{\Gamma}(K) = \varnothing$, then $K \cap \Gamma(K) \subseteq Int_{\Im_{\theta}}(K)$.*
(g) *$Bd^{\Gamma}(K) = \Gamma(Z \setminus K) \setminus \Psi_{\Gamma}(Z \setminus K) = Bd^{\Gamma}(Z \setminus K)$.*
(h) *$Z \setminus Bd^{\Gamma}(K) = \Psi_{\Gamma}(K) \cup \Psi_{\Gamma}(Z \setminus K)$.*
(i) *$Z = Bd^{\Gamma}(K) \cup \Psi_{\Gamma}(K) \cup \Psi_{\Gamma}(Z \setminus K) = Bd^{\Gamma}(Z \setminus K) \cup \Psi_{\Gamma}(K) \cup \Psi_{\Gamma}(Z \setminus K)$.*

Proof. **(a)** By the Theorem 2.2 *(i)*, $Bd^{\Gamma}(\varnothing) = \varnothing$.

(b) By the Theorem 2.2 *(i)*, $Bd^{\Gamma}(Z) = \varnothing$.

(c) If $K \in \Im$, then by the Theorem 2.2 *(ii)*, $Bd^{\Gamma}(K) = \varnothing \cap \Gamma(Z \setminus K) = \varnothing$.

(d) By the Theorem 2.2 *(iii)*, $Bd^{\Gamma}(K \cup L) = (\Gamma(K) \cup \Gamma(L)) \cap \Gamma((Z \setminus K) \cap (Z \setminus L))$. Then, from the Theorem 2.3, $Bd^{\Gamma}(K \cup L) \subseteq (\Gamma(K) \cup \Gamma(L)) \cap (\Gamma(Z \setminus K) \cap \Gamma(Z \setminus L)) = (\Gamma(K) \cap \Gamma(Z \setminus K) \cap \Gamma(Z \setminus L)) \cup (\Gamma(L) \cap \Gamma(Z \setminus K) \cap \Gamma(Z \setminus L)) \subseteq (\Gamma(K) \cap \Gamma(Z \setminus K)) \cup (\Gamma(L) \cap \Gamma(Z \setminus L)) = Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L)$.

(e) We know that $(K \cap Bd^{\Gamma}(L)) \cup Bd^{\Gamma}(K \cup L) \cup (L \cap Bd^{\Gamma}(K)) \subseteq Bd^{\Gamma}(L) \cup Bd^{\Gamma}(K \cup L) \cup Bd^{\Gamma}(K)$. From the Theorem 3.17 **(d)**, $Bd^{\Gamma}(L) \cup Bd^{\Gamma}(K \cup L) \cup Bd^{\Gamma}(K) \subseteq Bd^{\Gamma}(L) \cup Bd^{\Gamma}(K) = Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L)$. Thus, $(K \cap Bd^{\Gamma}(L)) \cup Bd^{\Gamma}(K \cup L) \cup (L \cap Bd^{\Gamma}(K)) \subseteq Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L)$.

(f) Let $Bd^{\Gamma}(K) = \varnothing$. Then, $\Gamma(K) \subseteq \Psi_{\Gamma}(K)$ by the Theorem 3.10. Assume that an element x of Z is not in $Int_{\Im_{\theta}}(K)$. Then $x \in cl_{\Im_{\theta}}(Z \setminus K) = (Z \setminus K) \cup \Gamma(Z \setminus K)$. If $x \in Z \setminus K$, then $x \notin K$ and so $x \notin K \cap \Gamma(K)$. If $x \in \Gamma(Z \setminus K)$, then $x \notin \Psi_{\Gamma}(K)$. Since $\Gamma(K) \subseteq \Psi_{\Gamma}(K)$, $x \notin \Gamma(K)$ and so $x \notin K \cap \Gamma(K)$. Therefore, we can say that: when $x \notin Int_{\Im_{\theta}}(K)$, $x \notin K \cap \Gamma(K)$ and so $K \cap \Gamma(K) \subseteq Int_{\Im_{\theta}}(K)$.

(g) By the Theorem 3.6, $Bd^{\Gamma}(Z \setminus K) = \Gamma(Z \setminus K) \setminus \Psi_{\Gamma}(Z \setminus K) = \Gamma(Z \setminus K) \setminus (Z \setminus \Gamma(Z \setminus (Z \setminus K))) = Bd^{\Gamma}(K)$.

(h) $Z \setminus Bd^{\Gamma}(K) = (Z \setminus \Gamma(K)) \cup (Z \setminus \Gamma(Z \setminus K)) = \Psi_{\Gamma}(Z \setminus K) \cup \Psi_{\Gamma}(K)$.

(i) $Z = (Z \setminus Bd^{\Gamma}(K)) \cup Bd^{\Gamma}(K)$. By the Theorem 3.17 **(h)**, $Z = \Psi_{\Gamma}(Z \setminus K) \cup \Psi_{\Gamma}(K) \cup Bd^{\Gamma}(K)$ and so $Z = \Psi_{\Gamma}(Z \setminus K) \cup \Psi_{\Gamma}(K) \cup Bd^{\Gamma}(Z \setminus K)$ from the Theorem 3.17 **(g)**. □

Remark 3.18. For subsets K, L of Z in $(Z, \tau, \mathfrak{I})$, although $Bd^{\Gamma}(K) = \emptyset$, K may not be in $\mathfrak{I}$. Furthermore, $Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L)$ may not be equivalent to $Bd^{\Gamma}(K \cup L)$. Similarly, $Bd^{\Gamma}(K) \cap Bd^{\Gamma}(L)$ may not be equivalent to $Bd^{\Gamma}(K \cap L)$.

Example 3.19. For $(Z, \tau, \mathfrak{I})$ in the Example 2.10, if $H = \{q, r, s\}$, then $Bd^{\Gamma}(H) = \emptyset$, but $H \notin \mathfrak{I}$. If $D = \{s\}$ and $L = \{q\}$, then $Bd^{\Gamma}(D \cup L) = \{p, q, r\}$, $Bd^{\Gamma}(D) = \{q, s\}$ and $Bd^{\Gamma}(L) = Z$, but $Bd^{\Gamma}(D) \cup Bd^{\Gamma}(L) \neq Bd^{\Gamma}(D \cup L)$. If $M = \{q, s\}$ and $N = \{r, s\}$, then $Bd^{\Gamma}(M \cap N) = \{q, s\}$, $Bd^{\Gamma}(M) = \{p, q, r\}$ and $Bd^{\Gamma}(N) = Z$, but $Bd^{\Gamma}(M) \cap Bd^{\Gamma}(N) \neq Bd^{\Gamma}(M \cap N)$.

Theorem 3.20. *$Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L) = Bd^{\Gamma}(K \setminus L) \cup Bd^{\Gamma}(K \cap L) \cup Bd^{\Gamma}(L \setminus K)$ for $K, L \subseteq Z$ in $(Z, \tau, \mathfrak{I})$.*

Proof. $(\Rightarrow)$: *(a)* By the Theorem 3.17 **(g)**, $Bd^{\Gamma}(K \cap L) = Bd^{\Gamma}(Z \setminus (K \cap L)) = Bd^{\Gamma}((Z \setminus K) \cup (Z \setminus L))$. Then by the Theorem 3.17 **(d)** and **(g)**, $Bd^{\Gamma}(K \cap L) = Bd^{\Gamma}((Z \setminus K) \cup (Z \setminus L)) \subseteq Bd^{\Gamma}(Z \setminus K) \cup Bd^{\Gamma}(Z \setminus L) = Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L)$.

(b) $Bd^{\Gamma}(K \setminus L) = Bd^{\Gamma}(K \cap (Z \setminus L)) = \Gamma(K \cap (Z \setminus L)) \cap \Gamma(Z \setminus [K \cap (Z \setminus L)]) = \Gamma(K \cap (Z \setminus L)) \cap \Gamma((Z \setminus K) \cup L)$. By the Theorem 2.3 and the Theorem 2.2 *(iii)*, $\Gamma(K \cap (Z \setminus L)) \cap \Gamma((Z \setminus K) \cup L) \subseteq (\Gamma(K) \cap \Gamma(Z \setminus L)) \cap (\Gamma(Z \setminus K) \cup \Gamma(L))$. Then $Bd^{\Gamma}(K \setminus L) \subseteq (\Gamma(K) \cap \Gamma(Z \setminus L)) \cap (\Gamma(Z \setminus K) \cup \Gamma(L)) = (\Gamma(K) \cap \Gamma(Z \setminus L) \cap \Gamma(Z \setminus K)) \cup (\Gamma(K) \cap \Gamma(Z \setminus L) \cap \Gamma(L)) \subseteq (\Gamma(K) \cap \Gamma(Z \setminus K)) \cup (\Gamma(Z \setminus L) \cap \Gamma(L)) = Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L)$.

(c) In a similar way to *(b)*, $Bd^{\Gamma}(L \setminus K) \subseteq Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L)$.

Hence, from *(a)*, *(b)* and *(c)*, $Bd^{\Gamma}(K \setminus L) \cup Bd^{\Gamma}(K \cap L) \cup Bd^{\Gamma}(L \setminus K) \subseteq Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L)$...(1)

$(\Leftarrow)$: $Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L) = Bd^{\Gamma}((K \setminus L) \cup (K \cap L)) \cup Bd^{\Gamma}((L \setminus K) \cup (K \cap L))$. Then by the Theorem 3.17 **(d)**, $Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L) \subseteq (Bd^{\Gamma}(K \setminus L) \cup Bd^{\Gamma}(K \cap L)) \cup (Bd^{\Gamma}(L \setminus K) \cup Bd^{\Gamma}(K \cap L))$. So $Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L) \subseteq Bd^{\Gamma}(K \setminus L) \cup Bd^{\Gamma}(L \setminus K) \cup Bd^{\Gamma}(K \cap L)$...(2).

Consequently, from (1) and (2), $Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L) = Bd^{\Gamma}(K \setminus L) \cup Bd^{\Gamma}(K \cap L) \cup Bd^{\Gamma}(L \setminus K)$. □

Theorem 3.21. *The following statements hold for $K, L \subseteq Z$ in $(Z, \tau, \mathfrak{I})$:*

(a) *$Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L) = Bd^{\Gamma}(K \cap L) \cup Bd^{\Gamma}(K \setminus L) \cup Bd^{\Gamma}(K \cup L)$.*

(b) *$Bd^{\Gamma}(K) \cup Bd^{\Gamma}(K \triangle L) = Bd^{\Gamma}(K \setminus L) \cup Bd^{\Gamma}(K \cap L) \cup Bd^{\Gamma}(L \setminus K)$.*

Proof. **(a)** $Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L) = Bd^{\Gamma}(K) \cup Bd^{\Gamma}(Z \setminus L)$ by the Theorem 3.17 **(g)**. From the Theorem 3.20, $Bd^{\Gamma}(K) \cup Bd^{\Gamma}(Z \setminus L) = Bd^{\Gamma}(K \setminus (Z \setminus L)) \cup Bd^{\Gamma}(K \cap (Z \setminus L)) \cup Bd^{\Gamma}((Z \setminus L) \setminus K)$. Then $Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L) = Bd^{\Gamma}(K \cap L) \cup Bd^{\Gamma}(K \setminus L) \cup Bd^{\Gamma}((Z \setminus L) \cap (Z \setminus K)) = Bd^{\Gamma}(K \cap L) \cup Bd^{\Gamma}(K \setminus L) \cup Bd^{\Gamma}(Z \setminus (K \cup L)) = Bd^{\Gamma}(K \cap L) \cup Bd^{\Gamma}(K \setminus L) \cup Bd^{\Gamma}(K \cup L)$ from the Theorem 3.17 **(g)**.

(b) $Bd^{\Gamma}(K) \cup Bd^{\Gamma}(K \triangle L) = Bd^{\Gamma}(K \setminus (K \triangle L)) \cup Bd^{\Gamma}((K \triangle L) \setminus K) \cup Bd^{\Gamma}(K \cap (K \triangle L))$ by the Theorem 3.20. Then $Bd^{\Gamma}(K) \cup Bd^{\Gamma}(K \triangle L) = Bd^{\Gamma}(K \setminus [(K \setminus L) \cup (L \setminus K)]) \cup Bd^{\Gamma}([(K \setminus L) \cup (L \setminus K)] \setminus K) \cup Bd^{\Gamma}(K \cap [(K \setminus L) \cup (L \setminus K)]) = Bd^{\Gamma}(K \cap [Z \setminus (K \setminus L)] \cap [Z \setminus (L \setminus K)]) \cup Bd^{\Gamma}([(K \setminus L) \cup (L \setminus K)] \cap (Z \setminus K)) \cup Bd^{\Gamma}([K \cap (K \setminus L)] \cup [K \cap (L \setminus K)]) = Bd^{\Gamma}(K \cap [(Z \setminus K) \cup L] \cap [(Z \setminus L) \cup K]) \cup Bd^{\Gamma}([(K \setminus L) \cap (Z \setminus K)] \cup [(L \setminus K) \cap (Z \setminus K)]) \cup Bd^{\Gamma}(K \setminus L) = Bd^{\Gamma}((K \cap L) \cap [(Z \setminus L) \cup K]) \cup Bd^{\Gamma}((L \setminus K) \cap (Z \setminus K)) \cup Bd^{\Gamma}(K \setminus L) = Bd^{\Gamma}(K \setminus L) \cup Bd^{\Gamma}(K \cap L) \cup Bd^{\Gamma}(L \setminus K)$. □

Corollary 3.22. *In $(Z, \tau, \mathfrak{I})$ for $K, L \subseteq Z$, $Bd^{\Gamma}(K) \cup Bd^{\Gamma}(L) = Bd^{\Gamma}(K \cap L) \cup Bd^{\Gamma}(K \setminus L) \cup Bd^{\Gamma}(K \cup L) = Bd^{\Gamma}(K) \cup Bd^{\Gamma}(K \triangle L) = Bd^{\Gamma}(K \setminus L) \cup Bd^{\Gamma}(K \cap L) \cup Bd^{\Gamma}(L \setminus K)$.*

Proof. It is clear by the Theorem 3.20 and Theorem 3.21. □

Theorem 3.23. *$Bd^{\Gamma}(K) = \Gamma(Z \setminus K)$ iff $Z \setminus \Gamma(K) \subseteq \Psi_{\Gamma}(K)$ for $K \subseteq Z$ in $(Z, \tau, \mathfrak{I})$.*

Proof. $Bd^{\Gamma}(K) = \Gamma(Z \setminus K) \Leftrightarrow \Gamma(Z \setminus K) \subseteq \Gamma(K) \Leftrightarrow Z \setminus \Gamma(K) \subseteq Z \setminus \Gamma(Z \setminus K) = \Psi_{\Gamma}(K)$. □

Theorem 3.24. *If K is an $\mathfrak{I}_{\Gamma}$-dense subset of Z, then $Bd^{\Gamma}(K) = \Gamma(Z \setminus K)$ in $(Z, \tau, \mathfrak{I})$.*

Proof. Let K be an $\mathfrak{I}_{\Gamma}$-dense subset of Z. Then, $\Gamma(K) = Z$. Thus, $Bd^{\Gamma}(K) = Z \cap \Gamma(Z \setminus K) = \Gamma(Z \setminus K)$. □

Remark 3.25. The reverse of the Theorem 3.24 may not be true in general.

Example 3.26. In the ideal topological space $(\mathbb{R}, P(\mathbb{R}), \mathfrak{I}_f)$, where $\mathfrak{I}_f$ is the ideal of finite subsets of $\mathbb{R}$, although $Bd^{\Gamma}(\mathbb{R}) = \emptyset = \Gamma(\mathbb{R} \setminus \mathbb{R})$, $\mathbb{R}$ is not $\mathfrak{I}_{\Gamma}$-dense.

Theorem 3.27. *In $(Z, \tau, \mathfrak{I})$, if K is an $\mathfrak{I}_{\theta}$-closed subset of Z, $Bd^{\Gamma}(K) \subseteq K \setminus \Psi_{\Gamma}(K)$.*

Proof. Let K be an $\mathfrak{I}_{\theta}$-closed subset of Z. Then, $\Gamma(K) \subseteq K$. Thus, $Bd^{\Gamma}(K) \subseteq K \cap \Gamma(Z \setminus K) = K \cap [Z \setminus (Z \setminus \Gamma(Z \setminus K))] = K \cap (Z \setminus \Psi_{\Gamma}(K)) = K \setminus \Psi_{\Gamma}(K)$. □

Remark 3.28. The reverse of the Theorem 3.27 may not be true in general.

Example 3.29. Let $Z = \{p, q, r, s\}, \mathfrak{I} = \{\emptyset, \{r\}\}$ and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Z\}$. In $(Z, \tau, \mathfrak{I})$, if $G = \{p, q, s\}$, $Bd^{\Gamma}(G) = \emptyset = G \setminus \Psi_{\Gamma}(G)$. Although $Bd^{\Gamma}(G) \subseteq G \setminus \Psi_{\Gamma}(G)$, the set G is not $\mathfrak{I}_{\theta}$-closed.

Corollary 3.30. *In $(Z, \tau, \mathfrak{I})$, if K is an $\mathfrak{I}_{\theta}$-closed subset of Z, $Bd^{\Gamma}(K) \subseteq K$.*

Proof. It is clear by the Theorem 3.27. □

Remark 3.31. The reverse of the Corollary 3.30 may not be true in general.

Example 3.32. In the Example 2.10, for $(Z, \tau, \mathfrak{I})$, if $H = \{q, r, s\}$, then $Bd^{\Gamma}(H) = \emptyset$ and $\Gamma(H) = Z$. Although $Bd^{\Gamma}(H) \subseteq H$, H is not $\mathfrak{I}_{\theta}$-closed.

Theorem 3.33. *If $Bd^{\Gamma}(K) \subseteq K$ and $\Psi_{\Gamma}(K) = \emptyset$, then K is $\mathfrak{I}_{\theta}$-closed for $K \subseteq Z$ in $(Z, \tau, \mathfrak{I})$.*

Proof. Suppose that $Bd^{\Gamma}(K) \subseteq K \subseteq Z$ and $\Psi_{\Gamma}(K) = \emptyset$. Then, $\Gamma(K) \setminus \Psi_{\Gamma}(K) \subseteq K$ by the Theorem 3.6. Therefore, $\Gamma(K) \setminus \emptyset = \Gamma(K) \subseteq K$. Thus, K is $\mathfrak{I}_{\theta}$-closed. □

Theorem 3.34. *If K is an $\mathfrak{I}_{\theta}$-open subset of Z, then $Bd^{\Gamma}(K) \subseteq \Gamma(K) \setminus K$ in $(Z, \tau, \mathfrak{I})$.*

Proof. Let K be an $\mathfrak{I}_{\theta}$-open subset of Z. Later on, by the Proposition 2.7 *(i)*, $Z \setminus K$ is $\mathfrak{I}_{\theta}$-closed. Hence $cl_{\mathfrak{I}_{\theta}}(Z \setminus K) = Z \setminus K$, that is, $(Z \setminus K) \cup \Gamma(Z \setminus K) = Z \setminus K$. It implies that $\Gamma(Z \setminus K) \subseteq Z \setminus K$. Thus, we can say that $Bd^{\Gamma}(K) \subseteq \Gamma(K) \cap (Z \setminus K) = \Gamma(K) \setminus K$. □

Remark 3.35. The reverse of the Theorem 3.34 may not be true in general.

Example 3.36. For $(Z, \tau, \mathfrak{I})$ in the Example 3.29, if $K = \{r\}$, then $Bd^{\Gamma}(K) = \emptyset \subseteq \Gamma(K) \setminus K$ but K is not $\mathfrak{I}_{\theta}$-open.

Theorem 3.37. *If K is $\mathfrak{I}_{\Gamma}$-dense and $Bd^{\Gamma}(K) \subseteq \Gamma(K) \setminus K$, then K is $\mathfrak{I}_{\theta}$-open for $K \subseteq Z$ in $(Z, \tau, \mathfrak{I})$.*

Proof. Suppose that K is $\mathfrak{I}_{\Gamma}$-dense subset of Z and $Bd^{\Gamma}(K) \subseteq \Gamma(K) \setminus K$. Then, $Bd^{\Gamma}(K) \subseteq \Gamma(K) \cap (Z \setminus K)$ and so $Z \setminus [\Gamma(K) \cap (Z \setminus K)] \subseteq Z \setminus (\Gamma(K) \cap \Gamma(Z \setminus K))$. Thus $(Z \setminus \Gamma(K)) \cup K \subseteq (Z \setminus \Gamma(K)) \cup (Z \setminus \Gamma(Z \setminus K)) = (Z \setminus \Gamma(K)) \cup \Psi_{\Gamma}(K)$. Then as K is $\mathfrak{I}_{\Gamma}$-dense, $(Z \setminus Z) \cup K \subseteq (Z \setminus Z) \cup \Psi_{\Gamma}(K)$. It implies that $K \subseteq \Psi_{\Gamma}(K)$ and so K is $\mathfrak{I}_{\theta}$-open by the Proposition 2.7 *(ii)*. □

Corollary 3.38. *For each θ-open subset U of Z in $(Z, \tau, \mathfrak{I})$, $Bd^{\Gamma}(U) \subseteq \Gamma(U) \setminus U$.*

Proof. Let U be a θ-open subset of Z. As $\tau_{\theta} \subseteq \sigma$, $U \in \sigma$ and so U is $\mathfrak{I}_{\theta}$-open by the Proposition 2.7 *(iii)*. Then, $Bd^{\Gamma}(U) \subseteq \Gamma(U) \setminus U$ from the Theorem 3.34. □

Corollary 3.39. *If K is both $\mathfrak{I}_{\theta}$-open and $\mathfrak{I}_{\theta}$-closed subset of Z, $Bd^{\Gamma}(K) = \emptyset$ in $(Z, \tau, \mathfrak{I})$.*

Proof. Assume that K is both $\mathfrak{I}_{\theta}$-open and $\mathfrak{I}_{\theta}$-closed subset of Z. Subsequently, $Bd^{\Gamma}(K) \subseteq \Gamma(K) \setminus K$ from the Theorem 3.34 and $Bd^{\Gamma}(K) \subseteq K \setminus \Psi_{\Gamma}(K)$ by the Theorem 3.27. Therefore, $Bd^{\Gamma}(K) \subseteq (\Gamma(K) \setminus K) \cap (K \setminus \Psi_{\Gamma}(K)) = \emptyset$ and so $Bd^{\Gamma}(K) = \emptyset$. □

Remark 3.40. In general, the reverse of the Corollary 3.39. may not be true. Look at the Example 3.29.

Theorem 3.41. *If K is $\mathfrak{I}_{\Gamma}$-perfect subset of Z, then $Bd^{\Gamma}(K) = K \setminus \Psi_{\Gamma}(K)$ in $(Z, \tau, \mathfrak{I})$.*

Proof. It is clear by the Theorem 3.6 and the definition of $\mathfrak{I}_{\Gamma}$-perfect set. □

Remark 3.42. The reverse of the Theorem 3.41 may not be true in general.

Example 3.43. For $(Z, \tau, \mathfrak{I})$ in the Example 2.10, if $H = \{q, r, s\}$, then $Bd^{\Gamma}(H) = \emptyset$, $H \setminus \Psi_{\Gamma}(H) = H \setminus Z = \emptyset$ and $\Gamma(H) = Z$. Although $Bd^{\Gamma}(H) = H \setminus \Psi_{\Gamma}(H)$, H is not $\mathfrak{I}_{\Gamma}$-perfect.

Theorem 3.44. *In $(Z, \tau, \mathfrak{I})$ for $K \subseteq Z$, if $Z \setminus K$ is $\mathfrak{I}_{\Gamma}$-dense and $Bd^{\Gamma}(K) = K \setminus \Psi_{\Gamma}(K)$, then K is $\mathfrak{I}_{\Gamma}$-perfect.*

Proof. Assume that $Z \setminus K$ is $\mathfrak{I}_{\Gamma}$-dense and $Bd^{\Gamma}(K) = K \setminus \Psi_{\Gamma}(K)$. Then $Z = \Gamma(Z \setminus K)$ and so $K \setminus \Psi_{\Gamma}(K) = K \cap \Gamma(Z \setminus K) = K \cap Z = K$. Moreover, $Bd^{\Gamma}(K) = \Gamma(K) \cap Z = \Gamma(K)$. $K = \Gamma(K)$, since $Bd^{\Gamma}(K) = K \setminus \Psi_{\Gamma}(K)$. Consequently, K is $\mathfrak{I}_{\Gamma}$-perfect. □

Theorem 3.45. *For $K \subseteq Z$, if $K \subseteq Bd^{\Gamma}(K)$, then K is Γ-dense-in-itself in $(Z, \tau, \mathfrak{I})$.*

Proof. The proof is clear. □

Remark 3.46. The inverse of the Theorem 3.45 may not be true.

Example 3.47. For $(Z, \tau, \mathfrak{I})$ in the Example 2.10, if $E = \{q, r\}$, then $Bd^{\Gamma}(E) = \{q, s\}$ and $\Gamma(E) = Z$. Although E is Γ-dense-in-itself, E is not a subset of $Bd^{\Gamma}(E)$.

4. The Relations of the Operator Bd^{Γ} with the Operator Bd and Bd^*

Definition 4.1 ([2]). In (Z, τ), the boundary operator $Bd : P(Z) \to \tau^k$ is defined as $Bd(K) = cl(K) \cap cl(Z \setminus K)$ for $K \subseteq Z$.

Definition 4.2 ([12]). In $(Z, \tau, \mathfrak{I})$, the operator $Bd^* : P(Z) \to \tau^k$ is defined as $Bd^*(K) = K^* \cap (Z \setminus K)^*$ for a subset K of Z and it is called $*$-boundary operator on $(Z, \tau, \mathfrak{I})$. If $x \in Bd^*(K)$, then the point x is called $*$-boundary point of K.

In [12], a new topology is obtained on Z by using $*$-boundary operator and it is shown that $k_1 : P(Z) \to P(Z)$, $k_1(K) = K \cup Bd^*(K)$ is a closure operator for this topology.

Theorem 4.3 ([2]). *In (Z, τ) for a subset K of Z;*
(i) *$x \in Bd(K)$ iff $x \in cl(K) \setminus int(K)$ for $x \in Z$.*
(ii) *$Bd(K) = \varnothing$ iff K is both open and closed.*

Remark 4.4. In (Z, τ), $Bd(Bd(K)) \subseteq Bd(K)$ for $K \subseteq Z$.

Theorem 4.5 ([12]). *$(Z, \tau, \mathfrak{I})$ is Hayashi-Samuel if and only if $Bd^*(K) = Bd(K)$ for each open subset K of Z.*

Remark 4.6. In an ideal topological space, the operator Bd^{Γ} may not provide the important properties of the boundary operator Bd. For example, the statements of $cl_{\mathfrak{I}_\theta}(K) \setminus Int_{\mathfrak{I}_\theta}(K) = Bd^{\Gamma}(K)$ and $Bd^{\Gamma}(Bd^{\Gamma}(K)) \subseteq Bd^{\Gamma}(K)$ may not be true in $(Z, \tau, \mathfrak{I})$ for a subset K of Z. Similarly, although $Bd^{\Gamma}(K) = \varnothing$, the set K may be neither $\mathfrak{I}_\theta$-open nor $\mathfrak{I}_\theta$-closed. Look at the Corollary 3.39 and the Example 3.29.

Example 4.7. For $(Z, \tau, \mathfrak{I}_1)$ in the Example 3.4, if $G = \{p, q, s\}$, then $Bd^{\Gamma}(G) = \varnothing$, but $cl_{\mathfrak{I}_\theta}(G) \setminus Int_{\mathfrak{I}_\theta}(G) = \{r\}$. Moreover, for $(Z, \tau, \mathfrak{I}_2)$ in the Example 3.4, if $C = \{r\}$, then $Bd^{\Gamma}(C) = \{p, q, r\}$ and $Bd^{\Gamma}(Bd^{\Gamma}(C)) = \{q, s\}$, but $Bd^{\Gamma}(Bd^{\Gamma}(C)) \nsubseteq Bd^{\Gamma}(C)$.

Remark 4.8. In an ideal topological space, there is not inclusion between Γ-boundary of a set and boundary of a set. For $(Z, \tau, \mathfrak{I})$ in the Example 2.10, $Bd^{\Gamma}(L) = Z$ and $Bd(L) = \{q\}$ for the set $L = \{q\}$. Similarly, for the set $K = \{p\}$ in this ideal topological space, $Bd^{\Gamma}(K) = \varnothing$ and $Bd(K) = \{p, q, r\}$. As a result, $Bd^{\Gamma}(L) \nsubseteq Bd(L)$ and $Bd(K) \nsubseteq Bd^{\Gamma}(K)$.

Theorem 4.9. *In $(Z, \tau, \mathfrak{I})$, $Bd^{\Gamma}(K) \subseteq Bd(K)$ for each θ-open subset K of Z.*

Proof. Let K be a θ-open subset of Z. Then, $Z \setminus K$ is θ-closed and thus K is an open set. By the Theorem 2.2 *(v)*, $Bd^{\Gamma}(K) \subseteq cl_\theta(K) \cap cl_\theta(Z \setminus K)$. From the Lemma 2.1 *(i)*, $Bd^{\Gamma}(K) \subseteq cl(K) \cap cl_\theta(Z \setminus K) = cl(K) \cap (Z \setminus K)$. Since $Z \setminus K$ is θ-closed, it is closed. So $Bd^{\Gamma}(K) \subseteq cl(K) \cap cl(Z \setminus K) = Bd(K)$. □

Theorem 4.10. *In $(Z, \tau, \mathfrak{I})$, $Bd^*(K) \subseteq Bd^{\Gamma}(K)$ for each subset K of Z.*

Proof. It is clear by the Lemma 2.1 *(ii)*. □

Remark 4.11. In $(Z, \tau, \mathfrak{I})$, $Bd^{\Gamma}(K)$ may not be a subset of $Bd^*(K)$ for a subset K of Z. For instance, for the ideal topological space in the Example 2.10, for the set $F = \{p, q\}$, $Bd^*(F) = \{q\}$ and $Bd^{\Gamma}(F) = Z$.

The collection of closed-discrete subsets $\mathfrak{I}_{cd}$, the collection of relatively compact subsets $\mathfrak{I}_k$, the collection of nowhere dense subsets $\mathfrak{I}_n$ and the collection of meager subsets $\mathfrak{I}_m$ are an ideal on Z for (Z, τ).

Theorem 4.12 ([11]). *In $(Z, \tau, \mathfrak{I})$, each of the following conditions implies, the local function and the local closure function are equivalent.*
(1) τ has a clopen base β.
(2) τ is T_3.
(3) $\mathfrak{I} = \mathfrak{I}_{cd}$.
(4) $\mathfrak{I} = \mathfrak{I}_k$.
(5) $\mathfrak{I}_n \subseteq \mathfrak{I}$.
(6) $\mathfrak{I} = \mathfrak{I}_m$.

Theorem 4.13 ([14]). *In $(Z, \tau, \mathfrak{I})$, each of the following conditions implies, the local function and the local closure function are equivalent.*

(1) τ has a clopen base β.
(2) τ is T_3.
(3) $\mathfrak{I} = \mathfrak{I}_{cd}$.
(4) $\mathfrak{I} = \mathfrak{I}_k$.
(5) $\mathfrak{I}_n \subseteq \mathfrak{I}$.
(6) $\mathfrak{I} = \mathfrak{I}_m$.
(7) Every open set is preclosed in (Z, τ).
(8) Every open set is closed in (Z, τ).
(9) Every open set is g-closed in (Z, τ).
(10) Every preopen set is closed in (Z, τ).

Corollary 4.14. *By the above theorem, each of the above conditions (1)-(10) implies $Bd^*(K) = Bd^\Gamma(K)$ for each $K \subseteq Z$ in $(Z, \tau, \mathfrak{I})$.*

Corollary 4.15. *Let $(Z, \tau, \mathfrak{I})$ be a Hayashi-Samuel space. In the Theorem 4.13, each of the conditions (1)-(10) implies $Bd^*(K) = Bd^\Gamma(K) = Bd(K)$ for each open subset K of Z.*

Proof. It is obvious by the Corollary 4.14 and the Theorem 4.5. □

5. New Operators

Definition 5.1. The operator $()^\Gamma_R : P(Z) \to P(Z)$ is defined as follows $K^\Gamma_R = \Gamma(K) \setminus K$ for $K \subseteq Z$ in $(Z, \tau, \mathfrak{I})$.

Theorem 5.2. *The following conditions hold for $K, L \subseteq Z$ in $(Z, \tau, \mathfrak{I})$.*

(a) $\varnothing^\Gamma_R = \varnothing$.
(b) $K \cap K^\Gamma_R = \varnothing$.
(c) $(K \cup L)^\Gamma_R = (K^\Gamma_R \setminus L) \cup (L^\Gamma_R \setminus K)$.
(d) $K^\Gamma_R \cup L^\Gamma_R = (K^\Gamma_R \cap L) \cup (K \cup L)^\Gamma_R \cup (K \cap L^\Gamma_R)$.

Proof. **(a)** $\varnothing^\Gamma_R = \Gamma(\varnothing) \setminus \varnothing = \varnothing$ by the Theorem 2.2 *(i)*.

(b) $K \cap K^\Gamma_R = K \cap (\Gamma(K) \setminus K) = \varnothing$.

(c) $(K \cup L)^\Gamma_R = (\Gamma(K) \cup \Gamma(L)) \cap [(Z \setminus K) \cap (Z \setminus L)]$ by the Theorem 2.2 *(iii)*. Then $(K \cup L)^\Gamma_R = [\Gamma(K) \cap (Z \setminus K) \cap (Z \setminus L)] \cup [\Gamma(L) \cap (Z \setminus K) \cap (Z \setminus L)] = [(\Gamma(K) \setminus K) \cap (Z \setminus L)] \cup [(\Gamma(L) \setminus L) \cap (Z \setminus K)] = (K^\Gamma_R \setminus L) \cup (L^\Gamma_R \setminus K)$.

(d) $(K \cap L^\Gamma_R) \cup (K \cup L)^\Gamma_R \cup (K^\Gamma_R \cap L) = [K \cap (\Gamma(L) \setminus L)] \cup [\Gamma(K \cup L) \setminus (K \cup L)] \cup [(\Gamma(K) \setminus K) \cap L]$. By the Theorem 2.2 *(iii)*, $(K \cap L^\Gamma_R) \cup (K \cup L)^\Gamma_R \cup (K^\Gamma_R \cap L) = [K \cap \Gamma(L) \cap (Z \setminus L)] \cup [(\Gamma(K) \cup \Gamma(L)) \cap (Z \setminus K) \cap (Z \setminus L)] \cup [\Gamma(K) \cap (Z \setminus K) \cap L] = [K \cap \Gamma(L) \cap (Z \setminus L)] \cup [\Gamma(K) \cap (Z \setminus K) \cap (Z \setminus L)] \cup [\Gamma(L) \cap (Z \setminus K) \cap (Z \setminus L)] \cup [\Gamma(K) \cap (Z \setminus K) \cap L] = ([\Gamma(L) \cap (Z \setminus L)] \cap [K \cup (Z \setminus K)]) \cup ([\Gamma(K) \cap (Z \setminus K)] \cap [(Z \setminus L) \cup L]) = [(\Gamma(L) \setminus L) \cap Z] \cup [(\Gamma(K) \setminus K) \cap Z] = (\Gamma(L) \setminus L) \cup (\Gamma(K) \setminus K) = K^\Gamma_R \cup L^\Gamma_R$. □

Theorem 5.3. *The following conditions hold for $K \subseteq Z$ in $(Z, \tau, \mathfrak{I})$.*

(a) *If K is $\mathfrak{I}_\theta$-open, then $Bd^\Gamma(K) \subseteq K^\Gamma_R$.*
(b) *If $Z \setminus K$ is $\mathfrak{I}_\Gamma$-perfect, then $Bd^\Gamma(K) = K^\Gamma_R$.*

Proof. **(a)** Let K be $\mathfrak{I}_\theta$-open. Then $Int_{\mathfrak{I}_\theta}(K) = Z \setminus cl_{\mathfrak{I}_\theta}(Z \setminus K) = K$. It implies that $cl_{\mathfrak{I}_\theta}(Z \setminus K) = Z \setminus K$ and so $\Gamma(Z \setminus K) \subseteq Z \setminus K$. Therefore, $Bd^\Gamma(K) \subseteq \Gamma(K) \setminus K = K^\Gamma_R$.

(b) Let $Z \setminus K$ be $\mathfrak{I}_\Gamma$-perfect. Then $\Gamma(Z \setminus K) = Z \setminus K$. It implies that $Bd^\Gamma(K) = \Gamma(K) \cap (Z \setminus K) = K^\Gamma_R$. □

Remark 5.4. The inverse of the above requirements may not be true.

Example 5.5. For $(Z, \tau, \mathfrak{I})$ in the Example 3.29, if $K = \{r\}$, then $Bd^\Gamma(K) = \varnothing \subseteq K^\Gamma_R$ but K is not $\mathfrak{I}_\theta$-open. Similarly, $Bd^\Gamma(K) = K^\Gamma_R = \varnothing$ but $Z \setminus K$ is not $\mathfrak{I}_\Gamma$-perfect.

Theorem 5.6. *A subset K of Z is $\mathfrak{I}_\theta$-closed iff $K^\Gamma_R = \varnothing$ in $(Z, \tau, \mathfrak{I})$.*

Proof. $(\Rightarrow)$: Assume that K is a $\mathfrak{I}_\theta$-closed subset of Z. In that case, $\Gamma(K) \subseteq K$. Thus, $K^\Gamma_R = \Gamma(K) \setminus K = \varnothing$.

$(\Leftarrow)$: Assume that $K^\Gamma_R = \varnothing$. Then, $\Gamma(K) \setminus K = \Gamma(K) \cap (Z \setminus K) = \varnothing$. Therefore, $(Z \setminus \Gamma(K)) \cup K = Z$ and hence $Z \setminus K \subseteq Z \setminus \Gamma(K)$. It implies that $\Gamma(K) \subseteq K$. Consequently, K is $\mathfrak{I}_\theta$-closed. □

Theorem 5.7. *If $K_R^\Gamma = Z$ for a subset K of Z, then K is $\Im_\Gamma$-dense in $(Z, \tau, \Im)$.*

Proof. Suppose that $K_R^\Gamma = Z$. Then $Z = \Gamma(K) \setminus K \subseteq \Gamma(K)$ and so $Z = \Gamma(K)$. As a result, K is $\Im_\Gamma$-dense. □

Remark 5.8. The inverse of the Theorem 5.7 may not be true in general.

Example 5.9. For $(Z, \tau, \Im)$ in the Example 2.10, if $L = \{q\}$, then L is an $\Im_\Gamma$-dense set but $L_R^\Gamma = \{p, r, s\} \neq Z$.

Definition 5.10. The operator $()^{\Gamma\Psi_\Gamma}$ on $(Z, \tau, \Im)$ is defined as: $K^{\Gamma\Psi_\Gamma} = K \setminus \Psi_\Gamma(K)$ for a subset K of Z.

Theorem 5.11. *The following conditions hold for $K, L \subseteq Z$ in $(Z, \tau, \Im)$.*
(a) $Z^{\Gamma\Psi_\Gamma} = \emptyset$.
(b) $K^{\Gamma\Psi_\Gamma} \subseteq K$.
(c) $(K \cap L)^{\Gamma\Psi_\Gamma} = (K^{\Gamma\Psi_\Gamma} \cap L) \cup (K \cap L^{\Gamma\Psi_\Gamma})$.
(d) $(K^{\Gamma\Psi_\Gamma})^{\Gamma\Psi_\Gamma} \subseteq K^{\Gamma\Psi_\Gamma}$.

Proof. **(a)** $Z^{\Gamma\Psi_\Gamma} = Z \setminus \Psi_\Gamma(Z) = Z \setminus (Z \setminus \Gamma(\emptyset)) = Z \setminus (Z \setminus \emptyset) = \emptyset$ by the Theorem 2.2 *(i)*.
(b) $K^{\Gamma\Psi_\Gamma} = K \setminus \Psi_\Gamma(K) \subseteq K$.
(c) $(K \cap L)^{\Gamma\Psi_\Gamma} = (K \cap L) \setminus (\Psi_\Gamma(K) \cap \Psi_\Gamma(L))$ by the Theorem 2.2 *(iv)*. Therefore, $(K \cap L)^{\Gamma\Psi_\Gamma} = (K \cap L) \cap [(Z \setminus \Psi_\Gamma(K)) \cup (Z \setminus \Psi_\Gamma(L))] = [K \cap L \cap (Z \setminus \Psi_\Gamma(K))] \cup [K \cap L \cap (Z \setminus \Psi_\Gamma(L))] = (K^{\Gamma\Psi_\Gamma} \cap L) \cup (K \cap L^{\Gamma\Psi_\Gamma})$.
(d) $(K^{\Gamma\Psi_\Gamma})^{\Gamma\Psi_\Gamma} = (K \setminus \Psi_\Gamma(K))^{\Gamma\Psi_\Gamma} = (K \setminus \Psi_\Gamma(K)) \setminus \Psi_\Gamma(K \setminus \Psi_\Gamma(K)) = (K \setminus \Psi_\Gamma(K)) \setminus \Psi_\Gamma(K \cap \Gamma(Z \setminus K))$. By the Theorem 2.2 *(iv)*, $(K^{\Gamma\Psi_\Gamma})^{\Gamma\Psi_\Gamma} = (K \setminus \Psi_\Gamma(K)) \setminus (\Psi_\Gamma(K) \cap \Psi_\Gamma(\Gamma(Z \setminus K))) = (K \cap \Gamma(Z \setminus K)) \cap [\Gamma(Z \setminus K) \cup (Z \setminus \Psi_\Gamma(\Gamma(Z \setminus K)))] = (K \cap \Gamma(Z \setminus K)) \cap [\Gamma(Z \setminus K) \cup \Gamma(Z \setminus \Gamma(Z \setminus K))] = [K \cap \Gamma(Z \setminus K) \cap \Gamma(Z \setminus K)] \cup [K \cap \Gamma(Z \setminus K) \cap \Gamma(Z \setminus \Gamma(Z \setminus K))] \subseteq K \cap \Gamma(Z \setminus K) = K \setminus \Psi_\Gamma(K) = K^{\Gamma\Psi_\Gamma}$. □

Theorem 5.12. *The following conditions hold for $K \subseteq Z$ in $(Z, \tau, \Im)$.*
(a) *K is R_Γ-perfect if and only if $K_R^\Gamma \in \Im$.*
(b) *If K is $\Im_\Gamma$-perfect, then $K_R^\Gamma = \emptyset$.*
(c) *If K is $\Im_\Gamma$-dense, then $K_R^\Gamma = Z \setminus K$.*
(d) *$Z \setminus K$ is R_Γ-perfect if and only if $K^{\Gamma\Psi_\Gamma} \in \Im$.*
(e) *If $Z \setminus K$ is $\Im_\Gamma$-perfect, then $K^{\Gamma\Psi_\Gamma} = \emptyset$.*
(f) *If $Z \setminus K$ is $\Im_\Gamma$-dense, then $K^{\Gamma\Psi_\Gamma} = K$.*

Proof. **(a)**, **(b)**, **(c)** The proofs are obvious.
(d) As $\Gamma(Z \setminus K) \setminus (Z \setminus K) = \Gamma(Z \setminus K) \cap K = K \setminus \Psi_\Gamma(K)$, the proof is obvious.
(e) Let $Z \setminus K$ be $\Im_\Gamma$-perfect. Then $Z \setminus K = \Gamma(Z \setminus K)$ and so $\Psi_\Gamma(K) = K$. Therefore, $K^{\Gamma\Psi_\Gamma} = \emptyset$.
(f) Let $Z \setminus K$ be $\Im_\Gamma$-dense. Then $Z = \Gamma(Z \setminus K)$ and so $K^{\Gamma\Psi_\Gamma} = K \setminus \Psi_\Gamma(K) = K \cap \Gamma(Z \setminus K) = K \cap Z = K$. □

Remark 5.13. In the above theorem, inverses of the requirements **(b)**, **(c)**, **(e)** and **(f)** may not be true in general.

Example 5.14. For $(Z, \tau, \Im)$ in the Example 2.10, if $K = \{p\}$, then $K_R^\Gamma = \emptyset$ but K is not $\Im_\Gamma$-perfect.

Example 5.15. In the ideal topological space $(\mathbb{R}, P(\mathbb{R}), \Im_f)$, although $\mathbb{R}_R^\Gamma = \emptyset = \mathbb{R} \setminus \mathbb{R}$, $\Gamma(\mathbb{R}) \neq \mathbb{R}$ and so $\mathbb{R}$ is not $\Im_\Gamma$-dense. Moreover, $\emptyset^{\Gamma\Psi_\Gamma} = \emptyset$ but $\mathbb{R} \setminus \emptyset = \mathbb{R}$ is neither $\Im_\Gamma$-perfect nor $\Im_\Gamma$-dense.

Authors Contribution Statement

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] Al-Omari, A., Noiri, T., *Local closure functions in ideal topological spaces*, Novi Sad J. Math., **43**(2)(2013), 139–149.
[2] Bourbaki, N., General Topology, Chapter 1-4, Springer, 1989.
[3] Dontchev, J., *Idealization of Ganster-Reilly decomposition theorems*, arXiv: math. Gn/9901017v1, 5 Jan 1999.
[4] Goyal, N., Noorie, N.S., *θ-closure and $T_{2\frac{1}{2}}$ spaces via ideals*, Italian Journal of Pure and Applied Mathematics, **41**(2019), 571–583.
[5] Kuratowski, K., Topology, Vol. I, Academic Press, New York, 1966.

[6] Levine, N., *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo, **19**(1970), 89–96.

[7] Mashhour, A.S., Abd El-Monsef, M.E., El-Deeb, S.N., *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt, **53**(1982), 47–53.

[8] Natkaniec, T., *On I-continuity and I-semicontinuity points*, Math. Slovaca, **36**(3)(1986), 297–312.

[9] Njamcul, A., Pavlović, A., *On closure compatibility of ideal topological spaces and idempotency of the local closure function*, Periodica Mathematica Hungarica, **84**(2)(2022), 221–234.

[10] Noorie, N.S., Goyal, N., *On* $S_{2\frac{1}{2}}$ *mod I spaces and* θ^{I}*-closed sets*, International Journal of Mathematics Trends and Technology, **52**(4)(2017), 226–228.

[11] Pavlović, A., *Local function versus local closure function in ideal topological spaces*, Filomat, **30**(14)(2016), 3725–3731.

[12] Selim, Sk., Modak, S., Islam, Md. M., *Characterizations of Hayashi-Samuel spaces via boundary points*, Commun. Adv. Math. Sci., **2**(3)(2019), 219–226.

[13] Tunç, A.N., Özen Yıldırım, S., *New sets obtained by local closure functions*, Annals of Pure and Applied Mathematical Sciences, **1**(1)(2021), 50–59.

[14] Tunç, A. N., Özen Yıldırım, S., *A study on further properties of local closure functions*, 7th International Conference on Recent Advances in Pure and Applied Mathematics (ICRAPAM 2020), (2020), 123–123.

[15] Vaidyanathaswamy, R., *The localisation theory in set-topology*, Proc. Indian Acad. Sci., Sect. A., **20**(1944), 51–61.

[16] Veličko, N. V., *H-closed topological spaces*, Mat. Sb. (N.S.), **70**(112)(1966), 98–112. English transl., Amer. Math. Soc. Transl., **78**(2)(1968), 102–118.