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# COLLOCATION METHOD APPLIED TO NUMERICAL SOLUTION OF INTEGRO-DIFFERENTIAL EQUATIONS

# İNTEGRO-DİFERANSİYEL DENKLEMLERİN SAYISAL ÇÖZÜMÜNE UYGULANAN KOLLOKASYON YÖNTEMİ

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## ABSTRACT

Integro-differential equations are encountered in such fields of study as mechanics, physics, chemistry, biophysics, astronomy, economic theory, and population dynamics. In rare cases the solution methods for differential and/or integral equations can be generalized to integro-differential equations; but in general, numerical methods have to be applied. Recent years have seen the development of a large number of methods applicable to integro-differential equations. The present study aims to compare these newer methods with the classical method of point collocation, which is one of the weighted residual methods. The method was applied to test problems chosen from the literature, both linear and nonlinear integro-differential equations, and was seen to give good results.

**Keywords:** Collocation method, nonlinear integro-differential equations, Volterra integro-differential equation, Fredholm integro-differential equations, numerical solution.

## ÖZET

İntegro-diferansiyel denklemler mekanik, fizik, kimya, biyofizik, astronomi, ekonomi teorisi ve nüfus dinamiği gibi çalışma alanlarında karşımıza çıkmaktadır. Nadir durumlarda diferansiyel ve/veya integral denklemlerin çözüm yöntemleri integro-diferansiyel denklemlere genelleştirilebilir; ancak genel olarak sayısal yöntemlerin uygulanması gerekir. Son yıllarda integro-diferansiyel denklemlere uygulanabilen çok sayıda yöntem geliştirilmiştir. Bu çalışma, bu yeni yöntemleri, ağırlıklı kalıntı yöntemlerinden biri olan klasik nokta kollokasyon yöntemi ile karşılaştırmayı amaçlamaktadır. Yöntem, literatürden seçilen doğrusal ve doğrusal olmayan integro-diferansiyel denklemlerden oluşan test problemlerine uygulanmış ve iyi sonuçlar verdiği görülmüştür.

**Anahtar Kelimeler:** Kollokasyon yöntemi, doğrusal olmayan integro-diferansiyel denklemler, Volterra integro-diferansiyel denklemi, Fredholm integro-diferansiyel denklemleri, sayısal çözüm.

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## INTRODUCTION

The first step in solving physics and engineering problems is the development of a physical-mathematical model of the problem. This leads to various types of equations which are classified according to the methods developed to solve them.

An integro-differential equation roughly is an equation that includes both the derivatives and the integrals of an unknown function u(x) to be solved. Integro-differential equations are encountered in a wide range of problems including physics and engineering, biomechanics, geophysics, electricity and magnetism, etc.

The works of Abel, Lotka, Fredholm, Malthus, Verhulst and Volterra developed general theories of integral and integro-differential equations (Lakshmikantham & Rao, 1995). However, the class of analytically solvable equations is quite limited and numerical solution methods often have to be applied. In recent years a number of novel numerical methods have been applied for the solution of integro-differential equations to model problems.

Some of these are: Modified Adomian Decomposition Method (MADM) (Olayiwola & Kareem, 2022), Finite Difference (Çakır & Güneş, 2022; Çimen & Enterili, 2020), Romberg extrapolation algorithm (REA) (Al-Towaiq & Kasasbeh, 2017), Parameterization Method (Dzhumabaev, 2016), Chebyshev Polynomials Method (Boonklurb et al., 2020, Sakran, 2019), Explicit Methods (Abdi, 2022), Multistep Runge-Kutta methods (Wen & Huang, 2024), Least squares method and the second kind Chebyshev wavelets (Ahmadinia et al., 2023).

In the next sections, test problems solved by these newer methods taken from literature will be solved by the collocation method and the results will be interpreted in terms of both effort and solution accuracy.

## MATERIAL AND METHODS

A general integro-differential equation involving a single-variable unknown function u(x) can be written as

$$u^{n}(x) = F\left[\, x, u(x), u'(x), \ldots, u^{(n-1)}(x) \,\right] + \int_{a}^{x} K\left[\, x, t, u(t), u'(t), \ldots, u^{(n)}(x) \,\right] dt \quad , a \geq 0 \tag{1}$$

F and K (called kernel) are given functions, and the equation is nonlinear if one of F or K is nonlinear. The type in which the upper limit of the integral is variable (the case here) is called a Volterra type equation. If the upper limit is a constant, this is called a Fredholm type equation.

The collocation method is widely applicable and stands out due to its simplicity. The main steps in applying the collocation method to any problem (including integro-differential equations) can be summarized as follows:

1. A polynomial form of approximate solution with unknown coefficients is suggested.
2. The approximate solution is substituted in the equation and the “residual” is evaluated.
3. Collocation points, including the boundary points, are chosen within the solution domain whose number totals the number of unknown coefficients.
4. The residual is evaluated at the collocation points and the results are equated to zero; this gives an algebraic system of equations.
5. Solving the algebraic system gives the unknown coefficients.

### *First example: Linear Volterra integro-differential equation*

The linear Volterra integro-differential equation

$$u'(x) = 1 - \int_{0}^{x} u(t)dt \ , \ \ x \geq 0 \quad u(0) = 0 \tag{2}$$

was solved using Improved Runge-Kutta Methods by Rabiei et al. (2019). For this problem the exact solution is $u(x) = \sin(x)$. Here, the approximate solution will be taken as

$$u(x) = \sum_{n=1}^{5} C_n x^n \tag{3}$$

Substituting (3) into (2), the residual is

$$R(x) = u'(x) - 1 + \int_0^x \left(\sum_{n=1}^{5} C_n t^n\right) dt \tag{4}$$

carrying out the derivative and integral gives

$$R(x) = -1 + C_1 + \frac{x^2 C_1}{2} + 2xC_2 + \frac{x^3 C_2}{3} + 3x^2 C_3 + \frac{x^4 C_3}{4} + 4x^3 C_4 + \frac{x^5 C_4}{5} + 5x^4 C_5 + \frac{x^6 C_5}{6} \tag{5}$$

There are N = 5 unknown coefficients; accordingly, the collocation points are chosen as.

$$x_i = \frac{1}{N+1} i \quad , (i = 1,2,3,4,5) \tag{6}$$

Residual is made zero at these points

$$R(x_i) = 0 \tag{7}$$

This gives a linear system of five equations for the unknown coefficients; in matrix form

$$\begin{bmatrix} 1.0041 & 0.1820 & 0.0248 & 0.0030 & 0.0003 \\ 1.0165 & 0.3656 & 0.099 & 0.0240 & 0.0054 \\ 1.0371 & 0.5522 & 0.2245 & 0.0814 & 0.0277 \\ 1.0661 & 0.7433 & 0.4010 & 0.1936 & 0.0878 \\ 1.1033 & 0.9403 & 0.6305 & 0.3795 & 0.2149 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{8}$$

and solving this system gives

$C_1$=0.9999986023769859,
$C_2$=0.000017054191493587497
$C_3$=-0.16676379652937337
$C_4$=0.0002715464065127452
$C_5$=0.008014287236481989

The approximate solution becomes

$$u(x) = 0.00801429x^5 + 0.000271546x^4 - 0.166764x^3 + 0.0000170542x^2 + 0.999999x \tag{9}$$

The solution was also found for N=10 and N=15 the exact solution together with approximate solutions and error values are presented in Table 1.

**Table 1.** Collocation solution and absolute errors for linear Volterra equation

| | | **Approximate solution values** | | | **Absolute Errors** | | |
|---|---|---|---|---|---|---|---|
| **x** | **Exact Solution** | **N=5** | **N=10** | **N=15** | **N=5** | **N=10** | **N=15** |
| 0.0 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.00000 | 0.00000 | 0.00000 |
| 0.1 | 0.099833 | 0.099806 | 0.099833 | 0.099833 | $2.66 \times 10^{-5}$ | $8.74 \times 10^{-13}$ | $9.06 \times 10^{-15}$ |
| 0.2 | 0.198669 | 0.198634 | 0.198669 | 0.198669 | $3.55 \times 10^{-5}$ | $8.44 \times 10^{-13}$ | $8.99 \times 10^{-15}$ |
| 0.3 | 0.295520 | 0.295483 | 0.295520 | 0.295520 | $3.68 \times 10^{-5}$ | $8.27 \times 10^{-13}$ | $8.82 \times 10^{-15}$ |
| 0.4 | 0.389418 | 0.389383 | 0.389418 | 0.389418 | $3.58 \times 10^{-5}$ | $7.98 \times 10^{-13}$ | $8.49 \times 10^{-15}$ |
| 0.5 | 0.479426 | 0.479391 | 0.479426 | 0.479426 | $3.43 \times 10^{-5}$ | $7.62 \times 10^{-13}$ | $8.16 \times 10^{-15}$ |
| 0.6 | 0.564642 | 0.564610 | 0.564642 | 0.564642 | $3.26 \times 10^{-5}$ | $7.19 \times 10^{-13}$ | $7.77 \times 10^{-15}$ |
| 0.7 | 0.644218 | 0.644187 | 0.644218 | 0.644218 | $3.06 \times 10^{-5}$ | $6.68 \times 10^{-13}$ | $7.21 \times 10^{-15}$ |
| 0.8 | 0.717356 | 0.717328 | 0.717356 | 0.717356 | $2.81 \times 10^{-5}$ | $6.13 \times 10^{-13}$ | $6.55 \times 10^{-15}$ |
| 0.9 | 0.783327 | 0.783301 | 0.783327 | 0.783327 | $2.56 \times 10^{-5}$ | $5.31 \times 10^{-13}$ | $5.99 \times 10^{-15}$ |
| 1.0 | 0.841471 | 0.841448 | 0.841471 | 0.841471 | $2.29 \times 10^{-5}$ | $1.29 \times 10^{-12}$ | $1.09 \times 10^{-14}$ |

***Second example: Linear Volterra integro-differential equation***

We consider the problem

$$u'(x) = \int_0^x x(1+2x)e^{t(x-t)}u(t)dt - u(x) + 1 + 2x$$
$$u(0) = 1\ ,\quad 0 \le x \le 1 \qquad (10)$$

Zarebnia (2010) suggested the Sinc collocation method to solve this problem. The exact solution is $e^{x^2}$. Noting that $u(0) = 1$, the approximate solution is taken as

$$u(x) = 1 + \sum_{n=1}^{10} C_n x^n \qquad (11)$$

Again, the residual is evaluated and set equal to zero at equally-spaced collocation points. The resulting approximate solutions for N=10, 15 and 20 are

$$u_{10}(x) = 0.0512657x^{10} - 0.152106x^9 + 0.29629x^8 - 0.254586x^7 + 0.33159x^6 - 0.0713121x^5 + 0.520591x^4 - 0.00387115x^3 + 1.00045x^2 - 0.000027284x + 1$$
$$u_{15}(x) = 0.000710335x^{15} - 0.0034912x^{14} + 0.0098353x^{13} - 0.0152694x^{12} + 0.0196096x^{11} - 0.00846689x^{10} + 0.0107384x^9 + 0.0364841x^8 + 0.00189345x^7 + 0.166146x^6 + 0.000106283x^5 + 0.499984x^4 + 1.60908\text{x}10^{-6}\,x^3 + x^2 + 3.97833\text{x}10^{-9}x + 1$$
$$u_{20}(x) = 4.66036\text{x}10^{-6}\,x^{20} - 0.0000241039x^{19} + 0.0000358925x^{18} + 0.000124398x^{17} - 0.000669626x^{16} + 0.0017409x^{15} - 0.000252207x^{14} + 0.0031339x^{13} - 0.00133798x^{12} + 0.00183359x^{11} + 0.0073663x^{10} + 0.000400867x^9 + 0.0415361x^8 + 0.0000331973x^7 + 0.16666x^6 + 9.63854\text{x}10^{-7}x^5 + 0.5x^4 + 7.99329\text{x}10^{-9}\,x^3 + x^2 + 1.12238\text{x}10^{-11}x + 1 \qquad (12)$$

Figure 1 compares approximate (N=10) and analytic solutions, and Table 2 gives the maximum errors in the approximate solutions.

**Table 2.** Maximum absolute errors for different collocation point selections of the solution of Eq. (10)

| **Collocation solutions** | | | |
|---|---|---|---|
| Sinc collocation Method (h=0.3141) | N = 10 | N = 15 | N=20 |
| $9.3\text{x}10^{-13}$ | $1\text{x}10^{-6}$ | $5\text{x}10^{-11}$ | $1\text{x}10^{-13}$ |

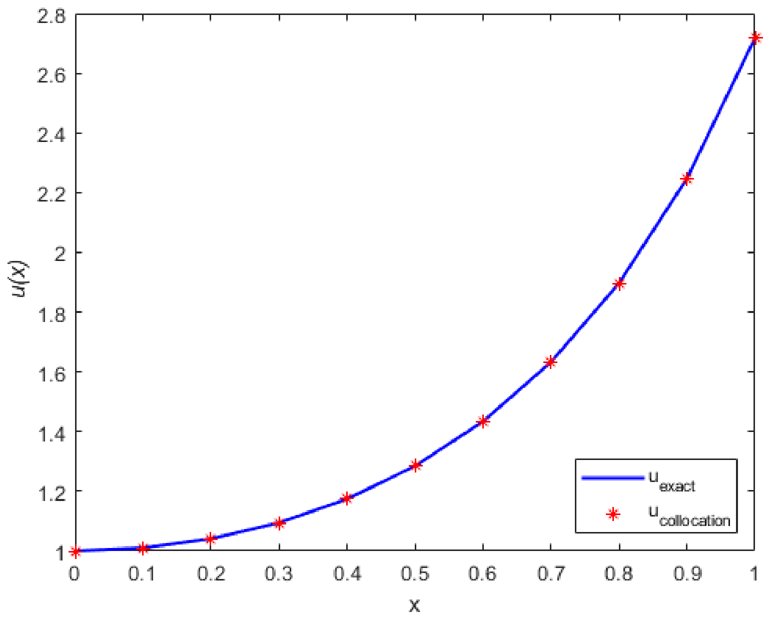


**Figure 1.** Analytical Solution and Approximate Solution for Ten Collocation Point of Lineer Volterra Integro-Differential Equation

### *Third example: Linear Fredholm integro-differential equation*

The problem

$$u'(x) = u(x) - \frac{1}{2}x + \frac{1}{x+1} - \text{In}\,(1+x) + \frac{1}{(\text{In}\,2)^2}\int_0^1 \frac{x}{1+t}u(t)dt$$
$$u(0) = 0 \tag{13}$$

was solved using Homotopy Analysis Method by Jaradat et al. (2008); the exact solution is $\text{In}\,(1+x)$. The approximate solution is taken as

$$u(x) = \sum_{n=1}^{10} C_n x^n \tag{14}$$

and this results in

$$u(x) = -0.00201698x^{10} + 0.0134619x^9 - 0.0427269x^8 + 0.0882951x^7 - 0.139456x^6 + 0.190119x^5 - 0.24747x^4 + 0.332896x^3 - 0.499952x^2 + 0.999997x \tag{15}$$

Table 3 gives absolute errors at various points within the solution domain.

**Table 3.** Absolute errors for approximate solution of Lineer Fredholm integro-differential equation

| | **Homotopy Analysis Method** | | | **Collocation** |
|---|---|---|---|---|
| x | $\lvert Ey_{HAM}^{10}\rvert$ | $\lvert Ey_{HAM}^{15}\rvert$ | $\lvert Ey_{HAM}^{20}\rvert$ | |
| 0.0 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 0.2 | 1.37x10$^{-5}$ | 1.59x10$^{-7}$ | 1.09x10$^{-8}$ | 9.09x10$^{-8}$ |
| 0.4 | 6.59x10$^{-5}$ | 7.64x10$^{-7}$ | 5.26x10$^{-8}$ | 1.25x10$^{-7}$ |
| 0.6 | 1.80x10$^{-4}$ | 2.09x10$^{-6}$ | 1.44x10$^{-7}$ | 1.78x10$^{-7}$ |
| 0.8 | 3.98x10$^{-4}$ | 4.61x10$^{-6}$ | 3.17x10$^{-7}$ | 2.51x10$^{-7}$ |
| 1.0 | 7.81x10$^{-4}$ | 9.05x10$^{-6}$ | 6.24x10$^{-7}$ | 3.85x10$^{-7}$ |

### *Fourth example: Linear Fredholm integro-differential equation*

The fourth problem is linear for which Xu (2007) used the variational iteration method for the solution.

$$u''(x) = e^x - \frac{4}{3}x + \int_0^1 xtu(t)dt$$
$$u(0) = 1 \ \ , \ u'(0) = 2 \tag{16}$$

The exact solution is $u(x) = x + e^x$. The approximate solution

$$u(x) = \sum_{n=0}^{10} C_n x^n \tag{17}$$

is modified as

$$u(x) = 1 + 2x + \sum_{n=2}^{10} C_n x^n \tag{18}$$

when the initial conditions in (16) are taken into account. Now, there are 9 unknown coefficients. Taking 9 equally-spaced collocation points gives

$$u(x) = 4.55861\text{x}10^{-7}x^{10} + 2.2831\text{x}10^{-6}x^9 + 0.0000254883x^8 + 0.000197785x^7 + 0.00138927x^6 + 0.00833318x^5 + 0.041667x^4 + 0.16667x^3 + 0.5x^2 + 2x + 1 \tag{19}$$

Table 4 shows the analytical and approximate solutions by collocation method and the absolute error.

**Table 4.** Absolute error values of collocation solution for Eq. (16)

| x | Analytical Results | Collocation Values | Absolute Error |
|---|---|---|---|
| 0.1 | 1.0000000000 | 1.00000000000 | $3.45\text{x}10^{-12}$ |
| 0.2 | 1.2051709180 | 1.20517091807 | $7.92\text{x}10^{-12}$ |
| 0.3 | 1.4214027581 | 1.42140275816 | $1.23\text{x}10^{-11}$ |
| 0.4 | 1.6498588075 | 1.64985880758 | $1.69\text{x}10^{-11}$ |
| 0.5 | 1.8918246976 | 1.89182469765 | $2.14\text{x}10^{-11}$ |
| 0.6 | 2.1487212707 | 2.14872127072 | $2.61\text{x}10^{-11}$ |
| 0.7 | 2.4221188003 | 2.42211880041 | $3.08\text{x}10^{-11}$ |
| 0.8 | 2.7137527074 | 2.71375270750 | $3.56\text{x}10^{-11}$ |
| 0.9 | 3.0255409284 | 3.02554092852 | $4.06\text{x}10^{-11}$ |
| 1.0 | 3.3596031111 | 3.35960311119 | $4.46\text{x}10^{-11}$ |

### *Fifth example: Linear Volterra-Fredholm equation*

The problem is

$$u'(x) = -2\sin(x) - x^2\sin(2x) + 2\sin(2x) - 2x\cos(2x) - 2e^x + 5e^{x-1} + 2x + \int_0^x \cos(t+x)\,u(t)dt + \int_0^1 \mathrm{e}^{\mathrm{x-t}}u(t)dt$$
$$u(0) = 0 \tag{20}$$

and it was solved by Rahmani et al. (2011) using Block Pulse Functions and Operational Matrices Method. Taking the approximate solution.

$$u(x) = \sum_{n=1}^{10} C_n x^n \tag{21}$$

gives the coefficients as

$$C_1 = -7.278\text{x}10^{-14} \quad C_6 = 2.9880\text{x}10^{-10}$$
$$C_2 = 1.0000 \quad C_7 = -4.018\text{x}10^{-10}$$
$$C_3 = -9.044\text{x}10^{-12} \quad C_8 = 3.3680\text{x}10^{-10} \qquad (22)$$
$$C_4 = 4.518\text{x}10^{-11} \quad C_9 = -1.598\text{x}10^{-10}$$
$$C_5 = -1.439\text{x}10^{-10} \quad C_{10} = 3.2730\text{x}10^{-11}$$

Ignoring the extremely small coefficients the approximate solution becomes $u(x) = x^2$, which is actually the exact solution. Here collocation points are distributed evenly between 0 and 1.

### *Sixth example: Nonlinear Volterra equation*

The problem is given as

$$u'(x) = -1 + \int_0^x u^2(t)dt \quad , \qquad x \geq 0$$
$$u(0) = 0 \qquad (23)$$

For this problem, Sepehrian & Razzaghiand (2004) suggested the single term Walsh series method, while Avudainayagam & Vani (2000) used the Wavelet-Galerkin method for the solution. Taking the approximate solution as

$$u(x) = \sum_{n=1}^{10} C_n x^n \qquad (24)$$

the residual is

$$R(x) = u'(x) + 1 - \int_0^x \left(\sum_{n=1}^{10} C_n t^n\right)^2 dt \qquad (25)$$

By calculating the coefficients $C_n$, the collocation solution is

$$u(x) = -0.00004x^{10} + 0.0007x^9 - 0.0012x^8 - 0.0027x^7 - 0.00075x^6 + 0.00032x^5 + 0.083x^4 + 0.000017x^3 - 2.004\text{x}10^{-6}x^2 - x \qquad (26)$$

Table 5 gives the comparison of the collocation results with the other two references. While finding an approximate solution, collocation points are distributed equally between 0 and 1.

**Table 5.** Comparison of the collocation method results with Wavelet-Galerkin Method and Walsh Series Method

| x | Exact Solution | Wavelet-Galerkin Method | Walsh Series Method | Collocation Method |
|---|---|---|---|---|
| 0.0000 | 0.000000 | 0.0000 | 0.00000 | 0.00000 |
| 0.0625 | -0.06250 | -0.0625 | -0.06250 | -0.06249 |
| 0.1250 | -0.12498 | -0.1250 | -0.12498 | -0.12498 |
| 0.1875 | -0.18740 | -0.1874 | -0.18740 | -0.18739 |
| 0.2500 | -0.24967 | -0.2497 | -0.24967 | -0.24967 |
| 0.3125 | -0.31171 | -0.3117 | -0.31171 | -0.31170 |
| 0.3750 | -0.37336 | -0.3734 | -0.37336 | -0.37335 |
| 0.4375 | -0.43446 | -0.4345 | -0.43446 | -0.43445 |
| 0.5000 | -0.49482 | -0.4948 | -0.49482 | -0.49482 |
| 0.5625 | -0.55423 | -0.5542 | -0.55423 | -0.55422 |
| 0.6250 | -0.61243 | -0.6124 | -0.61243 | -0.61243 |
| 0.6875 | -0.66917 | -0.6692 | -0.66916 | -0.66916 |
| 0.7500 | -0.72415 | -0.7242 | -0.72415 | -0.72415 |
| 0.8125 | -0.77709 | -0.7771 | -0.77709 | -0.77709 |
| 0.8750 | -0.82767 | -0.8277 | -0.82766 | -0.82766 |
| 0.9375 | -0.87557 | -0.8756 | -0.87557 | -0.87556 |
| 1.0000 | -0.92048 | -0.9205 | -0.92047 | -0.92047 |

In this problem if five collocation points are used, surprisingly, multiple solution sets are found including complex ones. Our impression is that taking the real solutions with the least absolute values gives both satisfactory results and a method of choosing among the many solutions. Table 6 shows the estimated solution values obtained by taking the real solution: $C_1 = 138.918, C_2 = -877.202, C_3 = 2594.84, C_4 = -3445.43, C_5 = 1786.8$ and the complex solution $C_1 = -3109.7 - 4719.65i, C_2 = 31682.5 + 59409.17\text{i}, C_3 = -127778.8 - 228312.45\text{i}, C_4 = 214035.32 + 341606.73\text{i}, C_5 = -123282.4 - 173256.87\text{i}$

**Table 6.** Values of approximate solutions for different real or complex coefficients

| x | Exact Solution | Real Coefficients | Complex Coefficients |
|---|---|---|---|
| 0.0000 | 0.000000 | 0.00000 | 0.0000 |
| 0.0625 | -0.06250 | 5.83844 | -98.6-113.6i |
| 0.1250 | -0.12498 | 7.93987 | -94.7-29.4i |
| 0.1875 | -0.18740 | 8.46831 | -75.5+80.7i |
| 0.2500 | -0.24967 | 8.73501 | -78.1+130.9i |
| 0.3125 | -0.31171 | 9.40295 | -103.4+100.7i |
| 0.3750 | -0.37336 | 10.6913 | -130.7+15.2i |
| 0.4375 | -0.43446 | 12.5800 | -131.03-74.3i |
| 0.5000 | -0.49482 | 15.0142 | -81.9-110.4i |
| 0.5625 | -0.55423 | 18.1086 | 18.6 -49.5i |
| 0.6250 | -0.61243 | 22.3523 | 138.4 +118.5i |
| 0.6875 | -0.66917 | 28.8130 | 196.5 +350.6i |
| 0.7500 | -0.72415 | 39.3416 | 49 +530.4i |
| 0.8125 | -0.77709 | 56.7767 | -524.2+448.4i |
| 0.8750 | -0.82767 | 85.1489 | -1834.9-217.4i |
| 0.9375 | -0.87557 | 129.886 | -4299.3-1922.1i |
| 1.0000 | -0.92048 | 198.016 | -8453.08-5273.07i |

### *Seventh example: High order Fredholm equation*

The third order problem

$$u'''(x) = sinx - x - \int_0^{\pi/2} xtu'(t)dt$$

$$u(0) = 1$$

$$u'(0) = 0$$

$$u''(0) = -1\,, \;\; 0 \leq x \leq \frac{\pi}{2} \tag{27}$$

was also studied by Al-Saar & Ghadle (2021) and the exact solution is $u(x) = \cos(x)$. Paying attention to the initial conditions in (27) approximate solution is taken as

$$u(x) = 1 - \frac{1}{2}x^2 + \sum_{n=3}^{10} C_n x^n \tag{28}$$

Absolute error values for various N values are given in Table 7.

**Table 7.** Absolute error values for collocation solutions

| x | Exact Solution | Collocation solution with N=15 | Absolute error for N=10 | Absolute error for N=15 |
|---|---|---|---|---|
| 0.00000 | 1.00000 | 1.00000 | 0.00000 | 0.00000. |
| 0.15708 | 0.987688 | 0.9876883 | $2.03 \times 10^{-10}$ | $1.30 \times 10^{-11}$ |
| 0.31415 | 0.951057 | 0.9510565 | $1.05 \times 10^{-9}$ | $2.08 \times 10^{-10}$ |
| 0.47123 | 0.891007 | 0.8910065 | $2.54 \times 10^{-9}$ | $1.05 \times 10^{-9}$ |
| 0.62831 | 0.809017 | 0.8090169 | $4.65 \times 10^{-9}$ | $3.33 \times 10^{-9}$ |
| 0.78539 | 0.707107 | 0.7071067 | $7.31 \times 10^{-9}$ | $8.13 \times 10^{-9}$ |
| 0.94247 | 0.587785 | 0.5877852 | $1.04 \times 10^{-8}$ | $1.68 \times 10^{-8}$ |
| 1.09956 | 0.453990 | 0.4539905 | $1.39 \times 10^{-8}$ | $3.12 \times 10^{-8}$ |
| 1.25664 | 0.309017 | 0.3090170 | $1.78 \times 10^{-8}$ | $5.32 \times 10^{-8}$ |
| 1.41372 | 0.156434 | 0.1564345 | $2.17 \times 10^{-8}$ | $8.53 \times 10^{-8}$ |
| 1.57080 | $6.12 \times 10^{-17}$ | $1.301 \times 10^{-7}$ | $2.58 \times 10^{-8}$ | $1.30 \times 10^{-7}$ |

### *Eighth example: Nonlinear Fredholm equation*

Finally, we use the same method to find the approximate solution of the nonlinear Fredholm integro-differential equation which is studied by Islam et al. (2013).

$$u'(x) + u(x) = \frac{1}{2}(e^{-2} - 1) + \int_0^1 u^2(t)dt$$

$$u(0) = 1 \tag{29}$$

Exact solution is $u(x) = e^{-x}$ . Approximate solution for (29) is taken as

$$u(x) = 1 + \sum_{n=1}^{10} C_n x^n \tag{30}$$

and after evaluating the coefficients $C_n$, the collocation solution is

$$u(x) = 1 - 0.999x + 0.499x^2 - 0.166x^3 + 0.041x^4 - 0.008x^5 + 0.001x^6 - 0.0001x^7 + 0.00002x^8 - 0.000002x^9 + 1.677 \times 10^{-7}x^{10} \quad (31)$$

Table 8 gives the errors in collocation as well as the two other methods indicated above.

**Table 8.** Comparison of absolute error values of the approximate solutions of Eq. (29)

| x | Haar wavelet | B-Spline wavelet | Collocation |
|---|---|---|---|
| 0.125 | 3.759x10$^{-7}$ | 7.5x10$^{-7}$ | 6.2983x10$^{-13}$ |
| 0.250 | 6.6413x10$^{-7}$ | 2.0x10$^{-7}$ | 6.4948x10$^{-13}$ |
| 0.375 | 8.6917x10$^{-7}$ | 1.8x10$^{-6}$ | 6.7090x10$^{-13}$ |
| 0.500 | 1.0020x10$^{-6}$ | 3.0x10$^{-6}$ | 6.9033x10$^{-13}$ |
| 0.625 | 1.0757x10$^{-6}$ | 3.4x10$^{-6}$ | 7.0754x10$^{-13}$ |
| 0.750 | 1.1029x10$^{-6}$ | 3.0x10$^{-6}$ | 7.2214x10$^{-13}$ |
| 0.875 | 1.0944x10$^{-6}$ | 1.6x10$^{-6}$ | 7.3024x10$^{-13}$ |

## CONCLUSION

Linear and nonlinear Volterra-Fredholm integro-differential equations were solved using the point collocation method. Eight test cases were chosen from the literature and the solutions were carried out using 5 to 20 collocation points. In all cases, it was observed that the solution is very close to exact solution for even 5 or 10 collocation points. The nonlinear problems naturally lead to a nonlinear system of algebraic equations which are considerably harder to solve than linear systems. It was observed that, of possible multiple solutions of the nonlinear system, the ones with the smallest absolute values give very good approximate solutions.

In terms of effort, the other solution methods mentioned above use complicated and computationally more costly mathematical operations compared to the collocation method which simply forms the residual and obtains a system of equations without any intermediary operations. Therefore, it is safe to conclude that the collocation method requires less effort.

These results show that the collocation is a powerful and simple to apply method which can easily be adopted to other types of equations such as fractional differential equations.

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