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Coefficient bounds for certain subclasses of analytic functions of complex order

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Abstract

In this paper, we introduce and investigate two subclasses of analytic functions of complex order, which are introduced here by means of a certain nonhomogeneous Cauchy–Euler-type differential equation of order m. Several corollaries and consequences of the main results are also considered.

Keywords: Analytic functions, Differential operator, Nonhomogeneous Cauchy-Euler differential equations, Coefficient bounds, Subordination.

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1. Introduction, definitions and preliminaries

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers,

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\}$$

be the set of positive integers and

$$\mathbb{N}^* := \mathbb{N} \setminus \{1\} = \{2, 3, 4, \ldots\}.$$

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{i=2}^{\infty} a_i z^i \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

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Recently, Faisal and Darus [8] defined the following differential operator:

$$
\begin{aligned}
D^0 f(z) &= f(z), \\
D^1_\lambda(\alpha,\beta,\mu) f(z) &= \left(\frac{\alpha-\mu+\beta-\lambda}{\alpha+\beta}\right) f(z) + \left(\frac{\mu+\lambda}{\alpha+\beta}\right) z f'(z), \\
D^2_\lambda(\alpha,\beta,\mu) f(z) &= D\left(D^1_\lambda(\alpha,\beta,\mu) f(z)\right) \\
&\vdots \\
D^n_\lambda(\alpha,\beta,\mu) f(z) &= D\left(D^{n-1}_\lambda(\alpha,\beta,\mu) f(z)\right). \qquad (1.2)
\end{aligned}
$$

If f is given by (1.1), then it is easily seen from (1.2) that

$$
D^n_\lambda(\alpha,\beta,\mu) f(z) = z + \sum_{i=2}^{\infty} \left(\frac{\alpha+(\mu+\lambda)(i-1)+\beta}{\alpha+\beta}\right)^n a_i z^i \tag{1.3}
$$

$$
\left(f \in \mathcal{A};\ \alpha,\beta,\mu,\lambda \geq 0;\ \alpha+\beta \neq 0;\ n \in \mathbb{N}_0\right).
$$

By using the operator $D^n_\lambda(\alpha,\beta,\mu)$, Faisal and Darus [8] defined a function class $\Psi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi)$ by

$$
\Re\left\{1+\frac{1}{\xi}\left(\frac{z\left[\zeta D^{n+1}_\lambda(\alpha,\beta,\mu) f(z)+(1-\zeta) D^n_\lambda(\alpha,\beta,\mu) f(z)\right]'}{\zeta D^{n+1}_\lambda(\alpha,\beta,\mu) f(z)+(1-\zeta) D^n_\lambda(\alpha,\beta,\mu) f(z)}-1\right)\right\} > \gamma,
$$

$$
\left(z \in \mathbb{U};\ 0 \leq \gamma < 1;\ 0 \leq \zeta \leq 1;\ \xi \in \mathbb{C}\setminus\{0\}\right)
$$

and also investigated the subclass $\Phi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi,\tau)$ of the analytic function class $\mathcal{A}$, which consists of functions $f \in \mathcal{A}$ satisfying the following nonhomogenous Cauchy-Euler differential equation:

$$
z^2\frac{d^2w}{dz^2}+2(1+\tau)z\frac{dw}{dz}+\tau(1+\tau)w=(1+\tau)(2+\tau)q(z)
$$

$$
\left(w=f(z) \in \mathcal{A};\ q \in \Psi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi);\ \tau \in (-1,\infty)\right).
$$

In the same paper [8], coefficient bounds for the subclasses $\Psi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi)$ and $\Phi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi,\tau)$ of analytic functions of complex order were obtained.

Making use of the differential operator $D^n_\lambda(\alpha,\beta,\mu)$, we now introduce each of the following subclasses of analytic functions.

1. Definition. Let $g : \mathbb{U} \to \mathbb{C}$ be a convex function such that

$$
g(0)=1 \qquad \text{and} \qquad \Re\{g(z)\} > 0 \quad (z \in \mathbb{U}).
$$

We denote by $\mathcal{M}_g(n,\alpha,\beta,\mu,\lambda,\zeta,\xi)$ the class of functions $f \in \mathcal{A}$ satisfying

$$
1+\frac{1}{\xi}\left(\frac{z\left[\zeta D^{n+1}_\lambda(\alpha,\beta,\mu) f(z)+(1-\zeta) D^n_\lambda(\alpha,\beta,\mu) f(z)\right]'}{\zeta D^{n+1}_\lambda(\alpha,\beta,\mu) f(z)+(1-\zeta) D^n_\lambda(\alpha,\beta,\mu) f(z)}-1\right) \in g(\mathbb{U}),
$$

where $z \in \mathbb{U};\ 0 \leq \zeta \leq 1;\ \xi \in \mathbb{C}\setminus\{0\}$.

2. Definition. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}_g(n,\alpha,\beta,\mu,\lambda,\zeta,\xi;m,\tau)$ if it satisfies the following nonhomogenous Cauchy-Euler differential equation:

$$
z^m\frac{d^mw}{dz^m}+\binom{m}{1}(\tau+m-1)z^{m-1}\frac{d^{m-1}w}{dz^{m-1}}+\cdots+\binom{m}{m}w\prod_{j=0}^{m-1}(\tau+j)=q(z)\prod_{j=0}^{m-1}(\tau+j+1)
$$

$$
\left(w=f(z) \in \mathcal{A};\ q \in \mathcal{M}_g(n,\alpha,\beta,\mu,\lambda,\zeta,\xi);\ m \in \mathbb{N}^*;\ \tau \in (-1,\infty)\right). \tag{1.4}
$$

Remark 1. There are many choices of the function g which would provide interesting subclasses of analytic functions of complex order. In particular,
(**i**) if we choose the function g as

$$g(z)=\frac{1+Az}{1+Bz}\quad\left(-1\leq B<A\leq 1;\ z\in\mathbb{U}\right),$$

it is easy to verify that g is a convex function in $\mathbb{U}$ and satisfies the hypotheses of Definition 1. If $f\in\mathcal{M}_g\left(n,\alpha,\beta,\mu,\lambda,\zeta,\xi\right)$, then we have

$$1+\frac{1}{\xi}\left(\frac{z\left[\zeta D_\lambda^{n+1}\left(\alpha,\beta,\mu\right)f\left(z\right)+\left(1-\zeta\right)D_\lambda^{n}\left(\alpha,\beta,\mu\right)f\left(z\right)\right]'}{\zeta D_\lambda^{n+1}\left(\alpha,\beta,\mu\right)f\left(z\right)+\left(1-\zeta\right)D_\lambda^{n}\left(\alpha,\beta,\mu\right)f\left(z\right)}-1\right)\prec\frac{1+Az}{1+Bz}\quad\left(z\in\mathbb{U}\right).$$

We denote this new class by $\mathcal{H}\left(n,\alpha,\beta,\mu,\lambda,\zeta,\xi,A,B\right)$. Also we denote by $\mathcal{B}\left(n,\alpha,\beta,\mu,\lambda,\zeta,\xi,A,B;m,\tau\right)$ for corresponding class to $\mathcal{M}_g\left(n,\alpha,\beta,\mu,\lambda,\zeta,\xi;m,\tau\right)$;
(**ii**) if we choose the function g as

$$g(z)=\frac{1+\left(1-2\gamma\right)z}{1-z}\quad\left(0\leq\gamma<1;\ z\in\mathbb{U}\right),$$

it is easy to verify that g is a convex function in $\mathbb{U}$ and satisfies the hypotheses of Definition 1. If $f\in\mathcal{M}_g\left(n,\alpha,\beta,\mu,\lambda,\zeta,\xi\right)$, then we have

$$\Re\left\{1+\frac{1}{\xi}\left(\frac{z\left[\zeta D_\lambda^{n+1}\left(\alpha,\beta,\mu\right)f\left(z\right)+\left(1-\zeta\right)D_\lambda^{n}\left(\alpha,\beta,\mu\right)f\left(z\right)\right]'}{\zeta D_\lambda^{n+1}\left(\alpha,\beta,\mu\right)f\left(z\right)+\left(1-\zeta\right)D_\lambda^{n}\left(\alpha,\beta,\mu\right)f\left(z\right)}-1\right)\right\}>\gamma\quad\left(z\in\mathbb{U}\right),$$

that is

$$f\in\Psi\left(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi\right).$$

Remark 2. In view of Remark 1(ii), by taking

$$g(z)=\frac{1+\left(1-2\gamma\right)z}{1-z}\quad\left(0\leq\gamma<1;\ z\in\mathbb{U}\right)$$

in Definitions 1 and 2, we easily observe that the function classes

$$\mathcal{M}_g\left(n,\alpha,\beta,\mu,\lambda,\zeta,\xi\right)\qquad\text{and}\qquad\mathcal{M}_g\left(n,\alpha,\beta,\mu,\lambda,\zeta,\xi;2,\tau\right)$$

become the aforementioned function classes

$$\Psi\left(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi\right)\qquad\text{and}\qquad\Phi\left(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi,\tau\right),$$

respectively.

In this work, by using the principle of subordination, we obtain coefficient bounds for functions in the subclasses

$$\mathcal{M}_g\left(n,\alpha,\beta,\mu,\lambda,\zeta,\xi\right)\qquad\text{and}\qquad\mathcal{M}_g\left(n,\alpha,\beta,\mu,\lambda,\zeta,\xi;m,\tau\right)$$

of analytic functions of complex order, which we have introduced here. Our results would unify and extend the corresponding results obtained earlier by Robertson [13], Nasr and Aouf [12], Altıntaş et al. [1], Faisal and Darus [8], Srivastava et al. [16], and others.

In our investigation, we shall make use of the principle of subordination between analytic functions, which is explained in Definition 3 below (see [11]).

3. Definition. For two functions f and g, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write

$$f(z)\prec g(z)\quad\left(z\in\mathbb{U}\right),$$

if there exists a Schwarz function $\mathfrak{w}(z)$, analytic in $\mathbb{U}$, with

$$\mathfrak{w}(0)=0\quad\text{and}\quad\left|\mathfrak{w}(z)\right|<1\quad\left(z\in\mathbb{U}\right),$$

such that

$$f(z) = g(\mathfrak{w}(z)) \quad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

2. Main results and their demonstration

In order to prove our main results (Theorems 1 and 2 below), we first recall the following lemma due to Rogosinski [14].

1. Lemma. Let the function $\mathfrak{g}$ given by

$$\mathfrak{g}(z) = \sum_{k=1}^{\infty} \mathfrak{b}_k z^k \qquad (z \in \mathbb{U})$$

be convex in $\mathbb{U}$. Also let the function $\mathfrak{f}$ given by

$$\mathfrak{f}(z) = \sum_{k=1}^{\infty} \mathfrak{a}_k z^k \qquad (z \in \mathbb{U})$$

be holomorphic in $\mathbb{U}$. If

$$\mathfrak{f}(z) \prec \mathfrak{g}(z) \qquad (z \in \mathbb{U}),$$

then

$$|\mathfrak{a}_k| \leq |\mathfrak{b}_1| \qquad (k \in \mathbb{N}).$$

We now state and prove each of our main results given by Theorems 1 and 2 below.

1. Theorem. Let the function $f \in \mathcal{A}$ be defined by (1.1). If the function f is in the class $\mathcal{M}_g(n, \alpha, \beta, \mu, \lambda, \zeta, \xi)$, then

$$|a_i| \leq \frac{(\alpha+\beta)^{n+1} \prod_{j=0}^{i-2} [j + |\xi| \, |g'(0)|]}{(i-1)! \, [\alpha + \zeta(\mu+\lambda)(i-1) + \beta] \, [\alpha + (\mu+\lambda)(i-1) + \beta]^n} \quad (i \in \mathbb{N}^*). \tag{2.1}$$

Proof. Let the function $f \in \mathcal{A}$ be given by (1.1). Suppose that the function $\mathcal{F}(z)$ is defined, in terms of the differential operator $D_\lambda^n(\alpha, \beta, \mu)$, by

$$\mathcal{F}(z) = \zeta D_\lambda^{n+1}(\alpha, \beta, \mu) f(z) + (1-\zeta) D_\lambda^n(\alpha, \beta, \mu) f(z) \quad (z \in \mathbb{U}). \tag{2.2}$$

Then, clearly, $\mathcal{F}$ is an analytic function in $\mathbb{U}$, and a simple computation shows that $\mathcal{F}$ has the following power series expansion:

$$\mathcal{F}(z) = z + \sum_{i=2}^{\infty} A_i z^i \quad (z \in \mathbb{U}), \tag{2.3}$$

where, for convenience,

$$A_i = \frac{[\alpha + \zeta(\mu+\lambda)(i-1) + \beta][\alpha + (\mu+\lambda)(i-1) + \beta]^n}{(\alpha+\beta)^{n+1}} a_i \quad (i \in \mathbb{N}^*). \tag{2.4}$$

From Definition 1 and (2.2), we thus have

$$1 + \frac{1}{\xi}\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} - 1\right) \in g(\mathbb{U}) \quad (z \in \mathbb{U}).$$

Let us define the function $p(z)$ by

$$p(z) = 1 + \frac{1}{\xi}\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} - 1\right) \quad (z \in \mathbb{U}). \tag{2.5}$$

Hence we deduce that

$$p(0)=g(0)=1 \quad \text{and} \quad p(z)\in g(\mathbb{U}) \quad (z\in\mathbb{U}).$$

Therefore, we have

$$p(z)\prec g(z) \quad (z\in\mathbb{U}).$$

Thus, according to the Lemma 1, we obtain

$$\left|\frac{p^{(l)}(0)}{l!}\right| \leq \left|g'(0)\right| \quad (l\in\mathbb{N}). \tag{2.6}$$

Also from (2.5), we find

$$z\mathcal{F}'(z)=\left[1+\xi\left(p(z)-1\right)\right]\mathcal{F}(z). \tag{2.7}$$

Next, we suppose that

$$p(z)=1+c_1z+c_2z^2+\cdots \quad (z\in\mathbb{U}). \tag{2.8}$$

Since $A_1=1$, in view of (2.3), (2.7) and (2.8), we obtain

$$(i-1)\,A_i=\xi\left\{c_{i-1}+c_{i-2}A_2+\cdots+c_1A_{i-1}\right\} \quad (i\in\mathbb{N}^*). \tag{2.9}$$

By combining (2.6) and (2.9), for $i=2,3,4$, we obtain

$$|A_2|\leq|\xi|\left|g'(0)\right|,$$

$$|A_3|\leq\frac{|\xi|\,|g'(0)|\,(1+|\xi|\,|g'(0)|)}{2!},$$

$$|A_4|\leq\frac{|\xi|\,|g'(0)|\,(1+|\xi|\,|g'(0)|)\,(2+|\xi|\,|g'(0)|)}{3!},$$

respectively. Also, by using the principle of mathematical induction, we obtain

$$|A_i|\leq\frac{\prod\limits_{j=0}^{i-2}\left[j+|\xi|\,|g'(0)|\right]}{(i-1)!} \quad (i\in\mathbb{N}^*).$$

Now from (2.4), it is clear that

$$|a_i|\leq\frac{(\alpha+\beta)^{n+1}\prod\limits_{j=0}^{i-2}\left[j+|\xi|\,|g'(0)|\right]}{(i-1)!\,\left[\alpha+\zeta\left(\mu+\lambda\right)(i-1)+\beta\right]\,\left[\alpha+\left(\mu+\lambda\right)(i-1)+\beta\right]^n} \quad (i\in\mathbb{N}^*).$$

This evidently completes the proof of Theorem 1. ∎

2. Theorem. Let the function $f\in\mathcal{A}$ be defined by (1.1). If the function f is in the class $\mathcal{M}_g\left(n,\alpha,\beta,\mu,\lambda,\zeta,\xi;m,\tau\right)$, then

$$|a_i|\leq\frac{(\alpha+\beta)^{n+1}\prod\limits_{j=0}^{i-2}\left[j+|\xi|\,|g'(0)|\right]\prod\limits_{j=0}^{m-1}(\tau+j+1)}{(i-1)!\,\left[\alpha+\zeta\left(\mu+\lambda\right)(i-1)+\beta\right]\,\left[\alpha+\left(\mu+\lambda\right)(i-1)+\beta\right]^n\prod\limits_{j=0}^{m-1}(\tau+j+i)} \quad (i\in\mathbb{N}^*).$$

Proof. Let the function $f\in\mathcal{A}$ be given by (1.1). Also let

$$h(z)=z+\sum_{i=2}^{\infty}b_iz^i\in\mathcal{M}_g\left(n,\alpha,\beta,\mu,\lambda,\zeta,\xi\right).$$

Hence, from (1.4), we deduce that

$$a_i = \frac{\prod_{j=0}^{m-1}(\tau+j+1)}{\prod_{j=0}^{m-1}(\tau+j+i)} b_i \quad (i \in \mathbb{N}^*,\ \tau \in (-1,\infty)).$$

Thus, by using Theorem 1, we obtain

$$|a_i| \le \frac{(\alpha+\beta)^{n+1}\prod_{j=0}^{i-2}[j+|\xi|\,|g'(0)|]}{(i-1)!\,[\alpha+\zeta(\mu+\lambda)(i-1)+\beta]\,[\alpha+(\mu+\lambda)(i-1)+\beta]^n}\,\frac{\prod_{j=0}^{m-1}(\tau+j+1)}{\prod_{j=0}^{m-1}(\tau+j+i)}.$$

This completes the proof of Theorem 2. ∎

3. Corollaries and consequences

In this section, we apply our main results (Theorems 1 and 2 of Section 2) in order to deduce each of the following corollaries and consequences.

1. Corollary. ([19]) Let the function $f \in \mathcal{A}$ be defined by (1.1). If the function f is in the class $\mathcal{M}_g(0,\alpha,\beta,\mu,\lambda,\zeta,\xi) \equiv \mathcal{S}_g(\zeta,\xi)$, then

$$|a_i| \le \frac{\prod_{j=0}^{i-2}[j+|\xi|\,|g'(0)|]}{(i-1)!\,(1+\zeta(i-1))} \quad (i \in \mathbb{N}^*).$$

2. Corollary. ([19]) Let the function $f \in \mathcal{A}$ be defined by (1.1). If the function f is in the class $\mathcal{M}_g(0,\alpha,\beta,\mu,\lambda,\zeta,\xi;m,\tau) \equiv \mathcal{K}_g(\zeta,\xi,m;\tau)$, then

$$|a_i| \le \frac{\prod_{j=0}^{i-2}[j+|\xi|\,|g'(0)|]}{(i-1)!\,(1+\zeta(i-1))}\,\frac{\prod_{j=0}^{m-1}(\tau+j+1)}{\prod_{j=0}^{m-1}(\tau+j+i)} \quad (i \in \mathbb{N}^*).$$

3. Corollary. ([17]) Let the function $f \in \mathcal{A}$ be defined by (1.1). If the function f is in the class $\mathcal{M}_g(n,1,0,0,1,\zeta,\xi) \equiv \mathcal{M}_g(n,\zeta,\xi)$, then

$$|a_i| \le \frac{\prod_{j=0}^{i-2}[j+|\xi|\,|g'(0)|]}{i^n\,(1+\zeta(i-1))\,(i-1)!} \quad (i \in \mathbb{N}^*).$$

4. Corollary. ([17]) Let the function $f \in \mathcal{A}$ be defined by (1.1). If the function f is in the class $\mathcal{M}_g(n,1,0,0,1,\zeta,\xi;2,\tau) \equiv \mathcal{M}_g(n,\zeta,\xi;\tau)$, then

$$|a_i| \le \frac{(1+\tau)(2+\tau)\prod_{j=0}^{i-2}[j+|\xi|\,|g'(0)|]}{i^n\,(1+\zeta(i-1))\,(i-1)!\,(i+\tau)\,(i+1+\tau)} \quad (i \in \mathbb{N}^*).$$

Setting

$$m=2 \quad \text{and} \quad g(z) = \frac{1+(1-2\gamma)z}{1-z} \quad (0 \le \gamma < 1;\ z \in \mathbb{U})$$

in Theorems 1 and 2, we have following corollaries, respectively.

5. Corollary. [8] Let the function $f \in \mathcal{A}$ be defined by (1.1). If the function f is in the class $\Psi(n, \alpha, \beta, \mu, \lambda, \zeta, \gamma, \xi)$, then

$$|a_i| \leq \frac{(\alpha+\beta)^{n+1} \prod_{j=0}^{i-2} [j + 2|\xi|(1-\gamma)]}{(i-1)!\ [\alpha + \zeta(\mu+\lambda)(i-1)+\beta]\ [\alpha + (\mu+\lambda)(i-1)+\beta]^n} \quad (i \in \mathbb{N}^*).$$

6. Corollary. [8] Let the function $f \in \mathcal{A}$ be defined by (1.1). If the function f is in the class $\Phi(n, \alpha, \beta, \mu, \lambda, \zeta, \gamma, \xi, \tau)$, then

$$|a_i| \leq \frac{(1+\tau)(2+\tau)(\alpha+\beta)^{n+1} \prod_{j=0}^{i-2} [j + 2|\xi|(1-\gamma)]}{(i+\tau)(i+1+\tau)(i-1)!\ [\alpha + \zeta(\mu+\lambda)(i-1)+\beta]\ [\alpha + (\mu+\lambda)(i-1)+\beta]^n} \quad (i \in \mathbb{N}^*).$$

For several other closely-related investigations, see (for example) the recent works [1-7, 12, 13].

References

[1] Altıntaş, O., Irmak, H., Owa, S. and Srivastava, H.M. *Coefficient bounds for some families of starlike and convex functions of complex order*, Appl. Math. Letters **20**, 1218-1222, 2007.

[2] Altıntaş, O., Irmak, H. and Srivastava, H.M. *Fractional calculus and certain starlike functions with negative coefficients*, Comput. Math. Appl. **30** (2), 9-15, 1995.

[3] Altıntaş, O. and Özkan, Ö. *Starlike, convex and close-to-convex functions of complex order*, Hacet. Bull. Nat. Sci. Eng. Ser. B **28**, 37-46, 1991.

[4] Altıntaş, O. and Özkan, Ö. *On the classes of starlike and convex functions of complex order*, Hacet. Bull. Nat. Sci. Eng. Ser. B **30**, 63-68, 2001.

[5] Altıntaş, O., Özkan, Ö. and Srivastava, H.M. *Neighborhoods of a class of analytic functions with negative coefficients*, Appl. Math. Letters **13** (3), 63-67, 1995.

[6] Altıntaş, O., Özkan, Ö. and Srivastava, H.M. *Majorization by starlike functions of complex order*, Complex Variables Theory Appl. **46**, 207-218, 2001.

[7] Altıntaş, O., Özkan, Ö. and Srivastava, H.M. *Neighborhoods of a certain family of multivalent functions with negative coefficient*, Comput. Math. Appl. **47**, 1667-1672, 2004.

[8] Faisal, I. and Darus, M. *Application of nonhomogenous Cauchy-Euler differential equation for certain class of analytic functions*, Hacet. J. Math. Stat. **43** (3), 375–382, 2014.

[9] Flett, T.M. *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl. **38**, 746-765, 1972.

[10] Jurg, I.B., Kim, Y.C. and Srivastava, H.M. *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl. **176**, 138-147, 1993.

[11] Miller, S.S. and Mocanu, P.T. *Differential subordinations*: Theory and applications. Monographs and Textbooks in Pure and Applied Mathematics, **225**. Marcel Dekker, Inc., New York, 2000.

[12] Nasr, M.A. and Aouf, M.K. *Radius of convexity for the class of starlike functions of complex order*, Bull. Fac. Sci. Assiut Univ. A **12** (1), 153-159, 1983.

[13] Robertson, M.S. *On the theory of univalent functions*, Ann. of Math. (2) **37** (2), 374-408, 1936.

[14] Rogosinski, W. *On the coefficients of subordinate functions*, Proc. London Math. Soc. (Ser. 2) **48**, 48-82, 1943.

[15] Sălăgean, G.Ş. *Subclasses of univalent functions*, Complex Analysis-Fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., vol. **1013**, Springer, Berlin, 1983, pp. 362-372.

[16] Srivastava, H.M., Altıntaş, O. and Serenbay, S.K., *Coefficient bounds for certain subclasses of starlike functions of complex order*, Appl. Math. Letters **24** (8), 1359-1363, 2011.

[17] Srivastava, H.M., Xu, Q.-H. and Wu, G.-P. *Coefficient estimates for certain subclasses of spiral-like functions of complex order*, Appl. Math. Letters **23**, 763-768, 2010.

[18] Uralegaddi, B.A. and Somanatha, C. *Certain classes of univalent functions.* Current topics in analytic function theory, 371-374, World Sci. Publ., River Edge, NJ, 1992.
[19] Xu, Q.-H., Gui, Y.-C. and Srivastava, H.M. *Coefficient estimates for certain subclasses of analytic functions of complex order*, Taiwanese J. Math. **15** (5), 2377-2386, 2011.